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INTRODUCTION TO PARTICLE PHYSICS

M.P. KHANNA

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INTRODUCTION TO PARTICLE PHYSICS
by M.P. Khanna

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To

My Mother, and Madhu, Vittu and Gaurav

for their blessings, encouragement, and support

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PREFACE

The kind of coherence in high energy physics that has emerged in the last few decades has never been witnessed before, primarily because of the all pervasive presence of gauge theories. No wonder, the particle physics texts give prominent place to gauge theories. However, these books—whether advanced or elementary—are not able to lay adequate emphasis on certain very important phenomenological developments.

Phenomenology has always played a crucial role as a link of experiment with theory. It is important, therefore, that students of high energy physics, both in experiment and theory, have enough exposure to the phenomenology of particle physics. With this view in mind, I have tried, in this book, to lay stress on phenomenology. I have discussed in considerable detail, topics such as symmetries, quark model, currents, concepts of gauge theories, the standard model, and flavour physics. I have focussed the readers' attention on the role that the currents have played in electromagnetic and weak interactions. Also, since nonleptonic weak decays have special place in weak interactions, I have covered them in a separate chapter. With some experimental data on heavy flavour physics being already available and with more to come in the near future, learning about heavy quark physics is timely for prospective high energy physicists. I have, therefore, given an adequate description of charm and beauty. Among the recent developments, the heavy quark effective theory for hadrons containing one heavy quark has been discussed.

The text is an outgrowth of the several lectures I have delivered to postgraduate and research students at Panjab University, Chandigarh for many years. It is also an outcome of a preparatory course. I gave at the SERC School on High Energy Physics held at the Physical Research Laboratory, Ahmedabad in 1991.

The framework needed to describe the properties of elementary particles involves relativity and quantum mechanics as the particles we are dealing with are very small and move fast. So the knowledge of relativistic quantum mechanics is assumed. However, after an introduction to the subject in Chapter 1, I give a brief review on relativistic quantum mechanics and quantum field theory in Chapter 2. Chapter 3 deals in detail with symmetry principles in general and discrete symmetries in particular, along with applications in Particle Physics. Chapters 4 and 5 cover unitary symmetry and quark model. Currents play a crucial role in describing electromagnetic and semi-leptonic weak interactions. In Chapter 6, I have attempted to show how currents are used in the description of electromagnetic and weak interactions of hadrons. Nonleptonic weak decays have a special place in weak interactions. So, I have discussed them separately in Chapter 7. Chapter 8 contains concepts of gauge invariance needed for the formulation of the standard model which is discussed in Chapter 9. Heavy flavour physics, a topic of great interest today, is presented in Chapter 10. In Chapter 11, I give some recent developments in the subject, in particular, an introduction to the recently introduced heavy flavour-spin symmetry and the heavy quark effective theory. I also introduce the link of the subject with cosmology.

The book is intended to serve as a one-semester/one-year course on Particle Physics at the postgraduate level or as an initial reading for research students in High Energy Physics.

I wish to express my appreciation to several students and colleagues of the Department, who have contributed to this work in many ways. I am, in particular, grateful to C.S. Aulukh, M.M. Gupta, V.S. Kaushal and R.C. Verma for valuable suggestions. Thanks are also due to S.C. Rawla for drawing the diagrams.

I am grateful to Professor A. Salam and Professor S. Randjbar-Daemi for the hospitality at the International Centre for Theoretical Physics, Trieste, Italy during my visits there in 1993 and 1995, which helped me in completing the manuscript.

M.P. KHANNA

1

INTRODUCTION

The past two decades have seen a remarkable progress in the physics of elementary particles. A host of new experimental results made possible by a new generation of particle accelerators and detectors and the convergence of theoretical ideas have brought to the subject a new coherence. Our current outlook has been shaped by the identification of quarks and leptons as the fundamental constituents of matter and by the gauge theory synthesis of the fundamental interactions.

The introduction is written in three parts:

1. Motivation of elementary particles.
2. Where do we stand [present status]?
3. How did we reach here [Historical Development]?

1.1 Motivation of Elementary Particle Physics

The world is full of diverse physical objects. It is a natural curiosity of man to ask, along with other questions, the following:

1. Are all the bewildering variety of objects that we see in nature made up of a small number of elementary (fundamental) particles?

2. If the answer to question (1) is in the affirmative, as it appears till now, then it is natural to ask how the fundamental particles cohere to make all the objects of the universe.

These questions are simple; however, the attempts to answer them need the help of (a) the most sophisticated and imaginative ideas and mathematical machinery as well as; (b) the most elaborate and most expensive scientific equipment.

The picture of fundamental constituents of matter and the interactions among them that has emerged in recent years is one of great beauty and simplicity. All matter seems to be composed of quarks and leptons, which are supposedly point-like (structureless), spin 1/2 particles. Leaving aside gravitation, which is a negligible perturbation at the energy scales usually considered, all the three interactions namely weak, electromagnetic, and strong, are described by gauge theories, and are mediated by spin one gauge quanta.

1.2 Present Status

1.2.1 Fundamental Constituents of Matter

There are three generations of leptons:

(i) ν_e, e, (ii) ν_μ, μ, and (iii) ν_τ, τ;

and three generations of quarks:

(i) u, d, (ii) s, c, and (iii) b, t.

All the six flavours u, d, s, c, b and t of quarks have been observed. Each flavour quark comes in three colours. Colour is just a quantum number like the charge and bears no similarity with the visual colour.

1.2.2 Mediators of Force

According to quantum field theory, the forces between the fundamental constituents of matter arise due to the exchange of gauge bosons. The mediators of the fundamental forces are: γ, W^+, W^-, Z, 8 coloured gluons. They are all spin 1 objects.

1.2.3 Higgs

Finally, there are Higgs particles (scalar spin 0 objects) which are supposed to be responsible for giving mass to the quarks, leptons and the intermediate vector bosons W^+, W^- and Z. They have not been seen so far.

1.2.4 Fundamental Interactions

All known processes in nature from microscopic to macroscopic (i.e. from sub-nuclear to extra galactic) can be understood as a manifestation of one or more of the four fundamental interactions: (a) gravitation, (b) electromagnetic, (c) weak, and (d) strong. These interactions have different strengths (see Table 1.1) and obey different conservation laws. A great success was made with the help of the principle of gauge invariance by the work of S.L. Glashow, S. Weinberg and A. Salam where they synthesise the electromagnetic and weak forces into a unified electroweak force. As a result, gauge symmetry assumed its role as

Table 1.1 Fundamental Interactions

Interaction	*Strength*	*Range*
Strong (colour)	$\alpha_S \approx 1$	Confinement
Strong (nuclear)	$g^2/4\pi \approx 14$	$m_\pi^{-1} \sim 1.5$ fm
Electromagnetic	$\alpha = \frac{1}{137}$	∞
Weak	$G_F = 1.026 \times 10^{-5}\ m_p^{-2}$	$M_W^{-1} \sim 10^{-3}$ fm
Gravitation	$G_N = M_{Pl}^{-2}$	∞

the guiding principle for the form of particle interactions. The strong interactions of quarks are described by Quantum Chromodynamics (QCD) which is a quantum field theory of the $SU(3)$ colour gauge group. The strong force is carried by $3^2 - 1 = 8$ gluons. The quarks and gluons are confined in hadrons and free quarks and gluons cannot be observed in nature. The picture is called the *standard model.*

1.3 Historical Development

The desire to identify the smallest constituents of matter is indeed very old. The first recorded studies are due to Greeks and Indians. The fact that the concept existed some thousands of years ago indicates that it was deep in the mind and philosophy even in those days to know what are the constituents of matter. But at that time the idea was merely speculative and philosophical. The philosophers at that time did not insist on seeing their atom. Modern scientific study of the subject, however, did not begin till the end of the nineteenth century with the discovery of electron by J.J. Thomson. In a way the study was started by J. Dalton in 1809 in his atomic theory. According to him each element consists of atoms, which he claimed are indivisible objects. But that the atom cannot hold the claim of being indivisible became clear in 1895, when J.J. Thomson showed that all atoms contain electrons. The electron is, therefore, an element of all atoms and hence of all substances. In addition to the electron, each atom consists of a nucleus which is located at the centre of the atom with most of its mass. The electrons and the nucleus of each atom are bound together by the Coulomb force.

In 1911, E. Rutherford showed that all nuclei contain protons (proton is the nucleus of Hydrogen atom). In 1932, J. Chadwick discovered the neutron (a particle with mass nearly equal to the mass of proton but with no electric charge) as a companion constituent of nuclei along with proton. Thus, all nuclei are made up of protons and neutrons. In the early thirties, therefore, it looked as if one aspect of particle physics, namely, the identification of elementary particles has been nearly understood: electron, proton and neutron being the constituents of all matter. What about the basic interactions among these particles?

The investigations of the gravitational interactions were initiated in 1666 by I. Newton. Coulomb's pioneering work on electromagnetic interactions appeared in 1776. Until 1930, almost all the phenomena observed in nature could be understood as a manifestation of these two classes of interactions. In 1930s, however, it appeared that there are forces in nature that are neither gravitational nor electromagnetic. In 1934, E. Fermi wrote down a beta nuclear decay Hamiltonian, which, with slight modification, is still believed to be the correct weak interaction Hamiltonian in the low energy limit. In 1935, H. Yukawa introduced yet another force, known as strong force responsible for binding together of the protons and neutrons inside the nuclei.

By the 1940s, therefore, the existence of four forces of nature was known. These forces—gravitational, electromagnetic, weak and strong—are still considered fundamental today, although we now recognize intimate connection between the weak and electromagnetic forces. The 1940s also saw the number of particles grow. The early twentieth century saw the birth of two major conceptual frameworks: relativity theory and quantum mechanics. These concepts were merged together by P.A.M. Dirac in 1928 resulting in the formulation of relativistic quantum mechanics and later together with Maxwell's theory by Schwinger,[1] Tomonaga,[2] Feynman[3] and F. Dyson in the development of Quantum Electrodynamics (QED) in 1947. In order to explain the solutions of the relativistic quantum mechanics, in particular, the Dirac equation, Dirac introduced the concept of antiparticle. He showed that to each particle there corresponds an antiparticle having the same mass as the particle but with opposite value of the additive quantum numbers like charge, lepton number, and strangeness. For example, to electron would correspond an antiparticle called *positron* which has the same mass as the electron and whose charge is equal in magnitude but opposite in sign to that of electron. The positron was observed by C.D. Anderson in 1932. The photon was known to be associated with the electromagnetic force. According to QED, the electromagnetic interactions take place through exchange of photon. W. Pauli in 1932 postulated the existence of neutrino to explain the continuous energy spectrum of nuclear beta decay. The discovery of two still more particles, now known as the muon and the pion, led to great excitement. The pion was necessary to account for the strong force, as predicted by Yukawa. The 1940s, 1950s and 1960s saw a proliferation of new particles discovered, using cosmic ray experiments at first, and later accelerators such as cyclotrons, betatrons and synchrotrons. Several new experimental techniques were also developed and a wealth of data collected. Many avenues of investigations were explored and many imaginative theoretical ideas introduced. New quantum numbers like strangeness and new particle symmetries like the isospin and $SU(3)$ were discovered during this period.

1.4 Particle Classification

All the observed particle states fall into two categories: (i) the leptons, which contain electron, muon and tau along with their associated neutrinos and do not participate in the strong interactions, and (ii) the hadrons, the strongly interacting particles.

1.4.1 Leptons

The electron was the first of the leptons, as a matter of fact, the first of the elementary particles, to be observed by J.J. Thomson in 1895. It is by far the lightest of all the charged particles. That is why it was named a lepton as lepton in Greek means light particle. The electron seems to have no structure but is known to have an intrinsic angular momentum (spin) of magnitude $\frac{1}{2}$.

Dirac's theory of electron associates with electron its antiparticle called positron with a mass equal to that of the electron but the value of all the additive quantum numbers, like charge, reversed. The positron was first discovered by C.D. Anderson in 1932 using a cloud chamber in magnetic field.

Electrons are emitted in β-decay and so experience the weak nuclear force. A simple example of β-decay is neutron decay to a proton and electron. If proton and electron were the only decay products, the electron energy will have a unique value because of the momentum and energy conservation. The observed electron energy spectrum is, however, continuous. To explain the continuous energy spectrum of electron, Pauli suggested that a neutral spin-$\frac{1}{2}$ massless (anti-electron neutrino $\bar{\nu}_e$) particle was also emitted in β-decay. The neutron decay thus is

---

[1] Schwinger, J. (1948): *Phys. Rev.*, *73*, 416.

[2] Tomonaga, S. (1948): *Phys. Rev.*, *74*, 224.

[3] Feynman, R.P. (1948): *Rev. Mod. Phys.*, *20*, 267.

$$n \rightarrow p + e^- + \bar{\nu}_e$$

Muon

The muon, the member of the second family (generation) of leptons, was discovered by Anderson and Neddermayer[1] in 1937 in a cloud chamber. Muons have mass about 200 times the mass of electron and carry charge equal to that of electron. Conversi, Pancini and Piccioni[2] in 1947 showed that both μ^- and μ^+ have the same lifetime of about 2×10^{-6} s. They separated positive and negative muons by their bending due to external magnetic field. The magnitude and equality of the lifetime of μ^- and μ^+ indicate that they do not interact with nuclei strongly and so could not be a Yukawa particles as had been thought earlier and so, like electrons, are leptons which do not participate in strong interactions. The muon decays through

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

The fact that the decay $\mu^- \rightarrow e^- + \gamma$ has never been observed indicates that muon is not simply a heavy electron. Obviously, muon differs from electron in type. Neutrinos associated with muon were seen in 1962. The reaction

$$\nu_\mu + n \rightarrow \mu^- + p$$

is seen whereas

$$\nu_\mu + n \rightarrow e^- + p$$

is not seen. The incident neutrinos in these reactions are taken from the decay of pion to muon and the accompanying neutrinos. The muon neutrinos are thus different from the electron neutrino. There should be separate conservation laws for the electron type lepton number (L_e) and muon type lepton number (L_μ).

Tau lepton

A yet heavier lepton was discovered in 1975 by Martin Perl and his collaborators, in the reaction

$$e^- + e^+ \rightarrow \tau^- + \tau^+$$

followed by the decays

$$\tau^- \rightarrow e^- + \bar{\nu}_e + \nu_\tau$$

and

$$\tau^+ \rightarrow \mu^+ + \nu_\mu + \bar{\nu}_\tau$$

---

[1] Anderson, C.D. and Neddermayer, S. (1937): *Phys. Rev.*, **51**, 884.

[2] Conversi, M., Pancini, E., and Piccioni, O. (1947): *Phys. Rev.*, **71**, 209.

The decays

$$\tau^- \to \mu^- + \mu^+ + \mu^-$$

and

$$\tau^- \to e^- + e^+ + e^-$$

are not seen, is an evidence of the fact that the τ-lepton is a different type from the electron and muon. Each lepton type number is separately conserved. The values of L_e, L_μ, and L_τ for the leptons are as follows:

L_e = +1 for e^- and ν_e, −1 for e^+ and $\bar{\nu}_e$, and 0 for all other leptons.

L_μ = +1 for μ^- and ν_μ, −1 for μ^+ and $\bar{\nu}_\mu$, and 0 for all other leptons.

L_τ = +1 for τ^- and ν_τ, −1 for τ^+ and $\bar{\nu}_\tau$, and 0 for all other leptons.

1.4.2 Hadronic States

The hadrons have a rich spectrum. There are a couple of hundred hadronic states and are further categorized into mesons and baryons. The mesons are integral spin particles and obey Bose-Einstein statistics whereas baryons have half odd-integral spins and are fermions. Some well-known hadron states and their properties are listed in Tables 1.2–1.4.

Table 1.2 Properties and Quark Content of Some Baryons
($J^P = 1/2^+$, $B = 1$)

Particle	*Mass* (GeV)	*I*	I_z	$Y = S + B$	*Quark content*
p	0.938	1/2	1/2	1	*uud*
n	0.940	1/2	−1/2	1	*udd*
Λ	1.116	0	0	0	*uds*
Σ^+	1.189	1	+1	0	*uus*
Σ^0	1.192	1	0	0	*uds*
Σ^-	1.197	1	−1	0	*dds*
Ξ^0	1.315	1/2	+1/2	−1	*uss*
Ξ^-	1.321	1/2	−1/2	−1	*dss*

The members of the very first hadron family, the proton and the neutron, have identical spin-parity, have nearly identical nuclear interactions and very nearly equal mass. They are classified using a quantum number called isospin, and denoted by *I*. This quantum number was introduced in 1932 by W. Heisenberg in analogy with spin. The proton and neutron are thought of as two different states of a particle which we call nucleon as these two are the constituents of nuclei. The nucleon is assigned isospin of 1/2. Proton and neutron are distinguished by their value of third component of isospin, the proton and neutron being the up and down states of nucleon and thus with $I_z = 1/2$ and $I_z = -1/2$ respectively. The π-mesons form triplet of isospin $I = 1$.

Table 1.3 Properties of Some Mesons

Particle *(spin 0)*	*Mass* (GeV)	I	I_z	$Y = S$	*Particle* *(spin 1)*	*Mass* (GeV)
π^+	0.140	1	+1	0	ρ^+	0.770
π^0	0.135	1	0	0	ρ^0	0.770
π^-	0.140	1	–1	0	ρ^-	0.770
K^+	0.494	1/2	+1/2	1	K^{*+}	0.892
K^0	0.498	1/2	–1/2	1	K^{*0}	0.897
$\bar{K}^0$	0.498	1/2	+1/2	–1	K^{*0}	0.897
K^-	0.494	1/2	–1/2	–1	K^{*-}	0.892
η	0.547	0	0	0	ω	0.783
η'	0.958	0	0	0	ϕ	1.020

Table 1.4 Properties and Quark Content of Some Baryon Resonances

$(J^P = 3/2^+, B = 1)$

Particle	*Mass* (GeV)	I	I_z	Y	*Quark content*
Δ^{++}	1.232	3/2	+3/2	1	*uuu*
Δ^+	1.232	3/2	+1/2	1	*uud*
Δ^0	1.232	3/2	–1/2	1	*udd*
Δ^-	1.232	3/2	– 3/2	1	*ddd*
Σ^{*+}	1.383	1	+1	0	*uus*
Σ^{*0}	1.385	1	0	0	*uds*
Σ^{*-}	1.387	1	–1	0	*dds*
Ξ^{*0}	1.532	1/2	+1/2	–1	*uss*
Ξ^{*-}	1.535	1/2	–1/2	–1	*dss*
Ω^-	1.672	0	0	–2	*sss*

Strangeness

With the advent of accelerators, proton beams produced more massive particles called strange particles. They are called strange because they are produced copiously in strong interactions yet decay through weak interactions. For instance, Λ^0 and K^0 are produced in

$$\pi^- + p \rightarrow \Lambda^0 + K^0 \tag{1.1}$$

and decay through

$$K^0 \rightarrow \pi^+ + \pi^- \quad \text{with } \tau = 0.89 \times 10^{-10} \text{ s}$$

$$\Lambda^0 \rightarrow p + \pi^- \quad \text{with } \tau = 2.63 \times 10^{-10} \text{ s}$$

The typical time scale by a strong interaction (production) process is of the order of 10^{-23} s. The paradox of very different times of production and decay is resolved by assigning to these particles a new additive quantum number, strangeness S, which is conserved in the strong production process and not conserved in the weak decay process. The earlier particles, the π-mesons and nucleon are assigned strangeness $S = 0$. For the new particles, we take $S(\Lambda^0) = -1$ and $S(K^0) = +1$ so that total strangeness in the final state of the production process is zero which is the value of the initial strangeness. However, the decay $\Lambda^0 \rightarrow p + \pi^-$ does not conserve strangeness and so is due to weak interaction. The same is the case with the decay of K^0. A quantum number hypercharge denoted by Y is defined as the sum of strangeness S and baryon number B.

Many more strange hadrons and resonances were produced and seen in cosmic rays in 1950s and 1960s. For example, we have a triplet Σ with mass of about 1195 MeV/c^2 and $S = -1$ and so $Y = 0$, a doublet of Ξ with mass of about 1315 MeV/c^2 with $S = -2$ and $Y = -1$. These are all baryons. Among the mesons with spin 0 and odd parity like the pions, we have a doublet of K (K^+, K^0) with $S = 1$, and so $Y = 1$ and their antiparticles a doublet of $\bar{K}$ ($\bar{K}^0$, K^-) with $S = -1$, and so $Y = -1$, a singlet η with $S = 0$ and $Y = 0$.

In addition, there are ten baryon resonances:

$$\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}; \quad \Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}; \quad \Xi^{*0}, \Xi^{*-}; \quad \Omega^-$$

with and $J = 3/2$ and positive parity. There are also nine meson resonances:

$$\rho^+, \rho^0, \rho^-; \quad K^{*+}, K^{*0}; \quad \bar{K}^{*0}, K^{*-}; \quad \omega, \phi$$

with spin 1 and negative parity which are called vector mesons. These resonant states live only for a short time of the order of 10^{-23} s, the characteristic time of strong interactions. They are low lying hadrons. As stated above, there are many more hadron states. All the known hadrons can be assigned to isospin multiplets with different values of Y. The charge of each hadron is given by the Gell-Mann Nishijima formula

$$Q = I_3 + \frac{Y}{2} \tag{1.2}$$

1.4.3 Symmetry and Quark Model

With the number of strongly interacting particles, the hadrons, increasing and none obviously more fundamental than any other, it was difficult to believe that they are all fundamental building blocks of nature, and so there were attempts to look for symmetries. Finally, in 1961 M. Gell-Mann and Y. Ne'eman proposed the eightfold-way, an $SU(3)$ symmetry scheme for the classification of hadrons, the baryons and the mesons. It was found that all observed baryons fall into the representations 1, 8 and 10 of the group $SU(3)$ of transformations whereas all mesons are grouped into the representations 1 and 8 of $SU(3)$. This observation served as a basis for the quark model, which was independently suggested by M. Gell-Mann and G. Zweig in 1964. According to the quark model, baryons are made up of three quarks while mesons are bound states of quark-antiquark pairs. Quarks are spin 1/2 particles and so obey Fermi-Dirac statistics. The properties of three light, also called ordinary, quarks u, d, s are listed in Table 1.5. The baryon wave function in terms of the quark wave functions, therefore, is to be completely antisymmetric. This requirement

Table 1.5 Quantum Numbers of the Three Quarks

Quark	B	I	I_3	Y	Q
u	1/3	1/2	+1/2	1/3	2/3
d	1/3	1/2	–1/2	1/3	–1/3
s	1/3	0	0	–2/3	–1/3

led O.W. Greenberg in 1964 to introduce an additional quantum number called colour to quarks. Later on more quarks, namely, charm, bottom and top had to be introduced and have been discovered.

Attempts were made by Yang and Mills[1] in 1954 to understand the strong interaction symmetry in analogy with the electromagnetic interactions via gauge theory, a concept which has now assumed its role as the guiding principle for the form of all particle interactions, although the Yang Mills theory was neglected for many years for its failure to understand strong interactions. Gauge symmetry was, however, rediscovered in the late 1960s and in about a decade became central to our understanding of the interactions of nature. The great success of this concept came due to the work of S.L. Glashow, S. Weinberg, and A. Salam where they, in 1967, synthesized the electromagnetic and weak forces into the unified electroweak force in a gauge theory. The discovery of neutral current, which the theory requires, in neutrino detectors provided the initial experimental confirmation of their model. The production and detection of the W and Z bosons at CERN in 1983 put the electroweak theory on a very firm footing. The electroweak forces are embedded in a direct product of a $SU(2)$ weak isospin gauge group with a $U(1)$ hypercharge gauge group and are governed by one coupling constant and a mixing angle. The gauge group dictates the boson particle spectrum, with four spin one gauge bosons: γ, Z, W^+ and W^-. These bosons are responsible for mediating the electroweak interactions between fermions.

With the success on the unification of the weak and electromagnetic interactions, gauge symmetry assumed its role as the guiding principle for the form of particle interaction in the standard model of the world, the strong, weak and electromagnetic forces are embedded in a $SU(3) \times SU(2) \times U(1)$ gauge group. The colour force is due to an $SU(3)$ colour gauge group with a strong coupling constant. The strong force is carried by 8 coloured gluons, which couple to colour fields. Because of the non-Abelian nature of the $SU(3)$ and $SU(2)$ gauge groups, the intermediate bosons, W's and gluons can have self-interactions, unlike the photon. This is due to the fact that the gauge bosons carry the same charge that they couple to, unlike Abelian gauge bosons such as the photon, which does not carry electric charge. Quarks are extraordinary in that they carry fractional electric charge. Hadrons, the physical particles which interact strongly, have no net colour charge.

1.4.4 Confinement

A paradoxical situation obtains, however. Quarks undoubtedly exist inside hadrons. The evidence for that comes not only from the quark systematics of hadrons but also by the probe of the interior of nucleons with high energy electrons and neutrinos. But it is impossible to see them loose. This is called quark confinement. Confinement is a feature of the strong interactions and states that the colour field can only exist in small regions of space, for example, inside hadrons. Quarks and gluons move about relatively freely inside hadrons with a force that becomes weaker, the closer they get. This feature of the theory is called the *asymptotic freedom*. However, the force grows as the quarks and gluons are separated. So, according to the $SU(3)$ theory of the strong interactions, free quarks and gluons cannot be observed in nature and they are only confined in hadrons.

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[1] Yang, C.N. and Mills, R.L. (1954); *Phys. Rev.*, **96**, 191.

1.4.5 Standard Model

From the preceding discussion the following picture emerges. Excluding gravitational forces, all the fundamental interactions of nature seem to share several common features:

1. They are all described by the same basic diagram (Fig. 1.1).

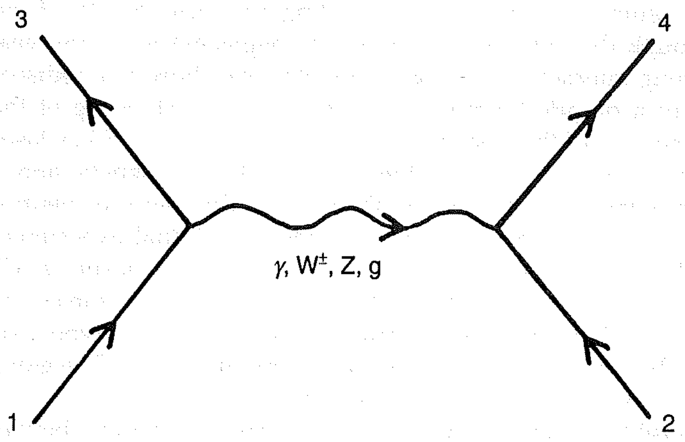

Fig. 1.1 Basic processes.

2. Each interaction has a gauge symmetry group; the strong $SU(3)_c$ and the electroweak $SU(2)_W \times U(1)_Y$.

So neglecting gravitation, the fundamental interactions correspond to a gauge symmetry $SU(3)_c \times SU(2)_W \times U(1)_Y \rightarrow SU(3)_c \times U(1)_{em}$, where the Higgs field breaks the electroweak gauge group but the colour and charge remain exactly conserved. This picture is called the *standard model*. Upto now all its predictions are in excellent agreement with experiment.

1.4.6 More Quarks

So far we have talked about the three quarks, namely *u*, *d*, *s*. The unified electroweak interactions needed neutral currents. Whereas the flavour conserving neutral currents were observed, the flavour changing neutral currents do not seem to exist in nature. In order to explain the absence of flavour changing neutral weak currents in the weak interactions, Glashow, Iliopoulos and Maiani[1] postulated in 1970 the existence of a fourth quark named charmed quark and labelled it *c*. Particles containing charm quark and antiquark (Quarkonium States) were observed in 1974. In 1975, a heavier lepton namely tau lepton was also observed. Since each lepton is to have a neutrino associated with it we have to have a ν_τ. Using lepton-quark symmetry, then Kobayashi and Maskawa[1] proposed that there should be six quarks and so introduced two more namely *b* and *t* quarks (Table 1.6). The particles consisting of *b* quark have also been observed and studied both experimentally and theoretically. In 1995, *t* quark was discovered, proving the existence of all the six quarks.

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[1] Glashow, S.L., Iliopoulos, J., and Maiani, L. (1970): *Phys. Rev.*, **D12**, 1285.

Table 1.6 Quantum Numbers of the Three Heavy Quarks

Quark	B	I	I_3	C	Q
c	1/3	0	0	1	2/3
b	1/3	0	0	0	–1/3
t	1/3	0	0	0	2/3

1.4.7 Beyond the Standard Model

However, the model does not explain several things:

- There are three gauge coupling constants. Can the number be reduced?
- Quarks and leptons both have spin 1/2. And leptonic and hadronic charged weak currents are identical in form. Are quarks and leptons related in any way? Why are

$$Q(e) + Q(p) = 0?$$
$$Q(\nu) - Q(e) = Q(u) - Q(d)?$$
$$Q(d) = (1/3)Q(e)?$$

This is charge quantization.

- How many generations of quarks and leptons are there?
- Standard model contains many parameters. Can this number be reduced?

In the framework of the Grand Unified Theories (GUTs), one may hope to find solution to some of these and other questions. In the simplest version of the GUTs the gauge group describing all interactions is $SU(5)$ which contains as its subgroup the gauge group $SU(3) \times SU(2) \times U(1)$. The quarks and leptons are put in the same representations of this group. There are, now, 24 gauge bosons and the 12 new ones can cause transition from a quark to a lepton and vice versa resulting in a possible decay of the proton.

Successful features of $SU(5)$ are:

- Quarks and leptons get related.
- Charged currents are $V - A$.
- Neutrino is massless.
- Charge is quantized.

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[1] Kobayashi, M. and Maskawa, T. (1973): *Prog. Theor. Phys.*, **49**, 652.

- $\sin^2 \theta = 0.2$ in approximate agreement with experiment.
- Proton decay is possible.
- It explains the matter antimatter asymmetry of the universe, thus bringing Particle Physics closer to Cosmology.
- Mass of the new intermediate vector boson (Lepto quark) can be large but below the Planck mass.

However, $SU(5)$ does not explain:

- Why do generations repeat?
- How many generations are there?
- Why do 12 orders of magnitude difference exist between two mass scales M_X, M_w?
- Gravity is not included.
- The number of parameters is still large.

1.4.8 Supersymmetry

The standard model and the GUTs are incomplete. There are still a large number of features unaccounted for: such as the generations of quarks and leptons and many of their properties which have to be put by hand. There is no relationship between the fundamental constituents (fermions) and the quanta (boson) of forces between them. On the other hand, the quantum theory of gravity is an unrenormalizable theory—a theory which offers escape from this difficulty at the same time achieving grand unification is a synthesis of general relativity with supersymmetry. Supersymmetry relates bosons with fermions. This is a symmetry which tells us that if we have a boson we must also have a fermion partner and vice versa. Thus, every fermion (quark or lepton) will have a supersymmetric (SUSY) spin zero partner (squark, slepton) while every boson (photon, gluon, W, Z, Higgs) has a spin 1/2 partner. The quantum of gravitational force, the graviton, which has spin 2 will have a supersymmetric partner gravitino with spin 3/2. In the supersymmetric limit, the particles and their supersymmetric partners should have equal masses. It is apparent that supersymmetry is badly broken as the supersymmetric partners have not been observed and so cannot have the same mass as ordinary particles and are probably very heavy. Even if all superparticles are heavy, the discovery of at least some of these particles appears to be quite feasible for the next generation of colliders. The discovery of superparticles would certainly be a triumph of symmetry ideas. Besides, that would definitely tell us a lot about the mechanism of symmetry breaking. Supersymmetry must then for the present be considered only to be an attractive hypothesis awaiting more data.

1.4.9 Superstring Theories

Recently, a far more ambitious attempt at unification that has conceptual similarity with general relativity has received a lot of attention. Called the Superstring theory, it proposes to combine the interactions of particle physics with gravity and is essentially a geometric theory. In contrast to conventional field theory, in which the elementary constituents are points in space time, superstring theory postulates that they are curves (or strings) in a certain D-dimensional curved space, where D is considerably larger than 4, with $D = 10$ being the value the current theory requires. The extra dimensions are not directly observable in macroscopic phenomena, however, because the *D*-dimensional space is supposed to curl up into *D*-4 compact dimensions, just as ordinary infinite cylinder defines 2-dimensional space with one compact dimension running around the cylinder and the other non-compact dimension being the cylinder.

1.5 Cosmology and Particle Physics

The grand unification scale being at energies which seem impossible to attain with accelerators, the theoretical physicists turned to Cosmology, and specifically to the first moments of the Big Bang. During the last couple of decades, therefore, close links have been established between particle physics and cosmology. The standard model of cosmology, the Big Bang model, relates to the fundamental questions in particle physics, in particular, the breaking of charge conjugation symmetry and baryon number conservation at GUT energies to account for the dominance of matter over antimatter in the universe; the existence of monopole; the existence of new species of particles and massive neutrinos. Three areas of recent activity at the interface of particle physics and cosmology are:

1. Nucleosynthesis and neutrino counting;
2. The dark matter problem; and
3. The formation of galaxies and large scale structure.

Both particle physics and cosmology have their standard models confirmed to a reasonable degree of accuracy by the present generation of experiments. While Hubble's work in the 1920s established the expanding universe, the two key quantitative observations: the microwave background and the Big Bang Nucleosynthesis (BBN) and light element abundances establish the modern physical cosmology and the hot big bang model. The second of these observations is more directly connected to high energy physics. Just as the new Cosmic Background Explorer (COBE) results have given renewed confidence in the 3 K background argument, the LEP collider has given us renewed confidence in the BBN argument.

Note that the microwave background probes events at temperatures of the order of 10^4 K and time of the order of 10^5 years, whereas the light abundances probe the universe at temperatures of the order of 10^{10} K and time of the order of 1 s. Thus, it is the nucleosynthesis results that have played the most significant role in leading to the particle physics-cosmology merger, that has taken place in the past decade or two.

2

RELATIVISTIC QUANTUM MECHANICS

2.1 Introduction

The framework to describe particles would require relativistic quantum mechanics and quantum field theory. In this chapter, we review essentials of these subjects. For more details the reader is referred to texts on relativistic quantum mechanics and introduction to quantum field theory. Let us begin with noting down the notations we are going to use. We shall follow the notation of Bjorken and Drell (1964). In natural units one takes $\hbar = c = 1$ and in relativistic quantum mechanics one uses natural units.

2.2 Relativistic Notations

The contravariant components of a four vector are indicated with a superscript. So,

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x}).$$

The covariant components of a vector will be indicated with a subscript:

$$x_\mu = g_{\mu\nu} x^\nu = (t, -\mathbf{x}),$$

the metric $g_{\mu\nu}$ being defined as

$$g_{00} = 1, \quad g_{kk} = -1 \quad (k = 1, 2, 3);$$
$$g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu,$$
$$g_{\mu\nu} = g^{\mu\nu}, \quad g^\mu_\nu = \delta^\mu_\nu .$$

The scalar product of x_μ with itself is given by

$$g^{\mu\nu} x_\mu x_\nu = x_\mu x^\mu = t^2 - \mathbf{x} \cdot \mathbf{x}$$

The gradient operator is

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right),$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right),$$

and the d'Alembertian operator is defined as

$$\Box \equiv \partial^{\mu}\partial_{\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2.$$

Fundamental tensors

In the four-dimensional Minkowski space, the totally antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is defined as

$$\varepsilon_{0ijk} = \varepsilon_{ijk},$$

where ε_{ijk} is the totally antisymmetric tensor (Ricci-Levi-Civita) in the three-dimensional Euclidean space and i, j, k run from 1 to 3.

$$\varepsilon_{0123} = 1,$$

and so

$$\varepsilon^{0123} = -1.$$

2.3 Relativistic Quantum Mechanics

2.3.1 Introduction

The simplest way of obtaining the Schrödinger equation in the non-relativistic quantum mechanics is to postulate the substitutions of derivative operators for momentum and energy. The substitutions

$$\mathbf{p} = -i\hbar\nabla, \qquad E = i\hbar\frac{\partial}{\partial t}, \tag{2.1}$$

in the energy-momentum relation

$$E = \frac{p^2}{2m}, \tag{2.2}$$

give the operator equation

$$H = -\frac{\hbar^2}{2m}\nabla^2, \tag{2.3}$$

for a free particle, where H is the operator for energy and

$$i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2. \tag{2.4}$$

These operators act on wave-function $\psi(\mathbf{r})$ and $\psi(\mathbf{r}, t)$, giving us the time independent wave equation

$$H\psi(\mathbf{r}) = E\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}), \tag{2.5}$$

and the time dependent wave equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) \tag{2.6}$$

respectively. In the presence of a potential $V(r)$, (2.6) takes the form

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2\,m}\nabla^2\psi(\mathbf{r},t) + V(r)\,\psi(\mathbf{r},t). \tag{2.7}$$

The non-relativistic quantum mechanics cannot solve the problems of the particles which are moving so fast that the non-relativistic energy momentum relation, (2.2) does not hold. Because of this, attempts were made from the very beginning of the development of quantum mechanics to formulate the relativistic form of the wave equation. The other limitation of the non-relativisitic quantum mechanics is that no material particle can be created or destroyed in this framework. That is, the number of particles remains constant with time. But in actual practice we know that particles are found to be created as well as destroyed. Both the limitations are not unrelated. When the particle is moving slowly, it has small kinetic energy and so cannot create other particles because of the energy conservation law. On the other hand, if the particle is moving with the velocity close to that of light, it has a lot of energy and so another particle can be created.

2.3.2 Klein-Gordon Equation

Free particle case

At the time when he was developing the non-relativistic wave equation, Schrödinger also suggested a relativistic generalization of it, which is known as Schrödinger's relativistic wave equation or more commonly the Klein-Gordon (KG) equation. We begin by considering the case of a free particle. The relativistic relation between energy E and momentum $\mathbf{p}$ of a particle with rest mass m is

$$E^2 = c^2\mathbf{p}^2 + m^2c^4. \tag{2.8}$$

Then, substituting (2.1), we get

$$-\hbar^2\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) = -\hbar^2c^2\nabla^2\phi(\mathbf{r},t) + m^2c^4\phi(\mathbf{r},t), \tag{2.9}$$

or

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{x},t) - \kappa^2\phi(\mathbf{x},t) = 0, \tag{2.10}$$

where

$$\kappa = \frac{mc}{\hbar}, \tag{2.11}$$

is the reciprocal of the Compton wavelength of the particle. In the natural units $\hbar = c = 1$ and so $\kappa = m$.

Difficulties of the Klein-Gordon equation

In the non-relativistic theory it is possible to define the probability density and probability current by employing the continuity equation. We follow the same method here starting with (2.10) and its conjugate equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi^* - \kappa^2\phi^* = 0. \tag{2.12}$$

Multiplying (2.10) on the left by ϕ^* and (2.12) on the right by ϕ and then subtracting, we obtain

$$\phi^*\nabla^2\phi - \phi\nabla^2\phi^* = \nabla \cdot \{\phi^*\nabla\phi - \phi\nabla\phi^*\} = \frac{1}{c^2}\left\{\phi^*\frac{\partial^2}{\partial t^2}\phi - \phi\frac{\partial^2}{\partial t^2}\phi^*\right\}. \tag{2.13}$$

This is in the form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{2.14}$$

In order to have the same expression for current as in the non-relativistic case, we multiply (2.14) by $\hbar/(2im)$ to get

$$\mathbf{j} = \frac{\hbar}{2im}(\phi^*\nabla\phi - \phi\nabla\phi^*), \tag{2.15}$$

$$\rho = -\frac{\hbar}{2imc^2}\left[\phi^*\frac{\partial}{\partial t}\phi - \left(\frac{\partial}{\partial t}\phi^*\right)\phi\right]. \tag{2.16}$$

The drawback of this expression is that there is no guarantee that ρ is positive definite and, therefore, it is not acceptable as a probability density. Because of this problem, the Klein-Gordon equation was first abandoned. In 1934, Pauli and Weisskopf gave the interpretation of (2.16) with ϕ as a field operator. According to them, (2.16), when multiplied by e, the charge of the particle, becomes acceptable as charge and current densities representing the many particle system.

2.3.3 The Dirac Equation

An important difference between the KG equation and the Schrödinger equation is that the time derivative is of the second order in the former, whereas it is of the first order in the latter. This means that while the Schrödinger equation when transformed into the equation of continuity gives probability density which is positive definite, the same procedure applied to the Klein-Gordon equation gives the expression for the probability density which is not positive definite as a probability density should be. Dirac noted that only first order equation would be free from the difficulty experienced in connection with the Klein-Gordon equation. He, therefore, started with the Hamiltonian operator H which should be linear in the components of momentum because the energy and momentum should be on the same footing due to relativity. This would ensure that the wave-equation is linear in time derivative as well as in space derivative, thereby preserving relativistic symmetry between space and time.

Free particle

Since the Hamiltonian should reduce to the rest mass energy when the particle momentum is zero, the simplest equation we can write is

$$\gamma^{\mu} p_{\mu} = m, \tag{2.17}$$

where we have introduced four quantities $\gamma^{\mu} = (\gamma^0, \boldsymbol{\gamma})$. Equation (2.17) on substitution from (2.1) gives

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0. \tag{2.18}$$

In order to find out what γ's should be, we note that ψ must satisfy the KG equation as every relativistic particle satisfies the energy momentum relation (2.8) on which the KG equation is based. We multiply (2.18) on the left by $(i\gamma^{\nu}\partial_{\nu} + m)$ and get

$$(i\gamma^{\nu}\partial_{\nu} + m)\,(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \tag{2.19}$$

or

$$(\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu} + m^2)\psi = 0. \tag{2.20}$$

Symmetrizing with respect to ν and μ we write (2.20) as

$$[\frac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu})\partial_{\mu}\partial_{\nu} + m^2]\psi = 0. \tag{2.21}$$

If γ's satisfy

$$\frac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}) = g^{\mu\nu}I, \tag{2.22}$$

then ψ will satisfy the KG equation. It is clear from (2.22) that γ's cannot be numbers. They are matrices. We can cast (2.18) in the form

$$i\frac{\partial}{\partial t}\psi = H\psi, \tag{2.23}$$

with

$$H = -i\gamma^0\boldsymbol{\gamma}\cdot\nabla + \gamma^0 m = \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + \gamma^0 m. \tag{2.24}$$

The hermiticity of the Hamiltonian H and the momentum $\mathbf{p}$ demands that γ^0 be Hermitian and $\boldsymbol{\gamma}$ be anti-Hermitian. Thus,

$$\gamma^{0\dagger} = \gamma^0\,, \qquad \gamma^{k\dagger} = -\gamma^k.$$

These hermiticity relations can be written as

$$(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0. \tag{2.25}$$

We shall raise or lower indices on γ's even though they are not components of a four vector by means of metric tensors. That is,

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu, \tag{2.26}$$

$$\gamma^\mu = g^{\mu\nu}\gamma_\nu. \tag{2.27}$$

It can be seen that γ's are $n \times n$ matrices with n even. For $n = 2$ the three Pauli matrices σ_i satisfy relations of the type (2.22). But then we cannot have a fourth 2×2 matrix which anti-commutes with all the three Pauli matrices. We, therefore, need at least 4×4 matrices. For the $4 \times 4\gamma$ matrices, we adopt the following representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{2.28}$$

Each element of these matrices is a 2×2 matrix. For the Dirac equation (2.18) to have any meaning then, ψ should have the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{2.29}$$

The Hermitian conjugate of the Dirac equation is

$$i\partial_\mu \psi^\dagger(x)(\gamma^\mu)^\dagger + m\psi^\dagger(x) = 0. \tag{2.30}$$

Defining an adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0, \tag{2.31}$$

we obtain the adjoint equation

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\psi = 0. \tag{2.32}$$

2.3.4 The Continuity Equation for the Dirac Equation

Probability and current density

Multiplying (2.18) on the left by the row matrix $\bar{\psi}$ and (2.32) on the right by the column matrix ψ and adding, we get

$$(\bar{\psi} i\gamma^\mu \partial_\mu \psi) + (i\partial_\mu \bar{\psi}\gamma^\mu)\psi = 0, \tag{2.33}$$

which can be written as

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0. \tag{2.34}$$

Equation (2.34) is a continuity equation with the current density j^μ given by

$$j^\mu = (\bar{\psi}\gamma^\mu\psi) \equiv (\rho, \mathbf{j}). \tag{2.35}$$

Thus

$$\rho = \psi^{\dagger}\psi = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

$$= (\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4) = \sum_i |\psi_i|^2, \tag{2.36}$$

is positive definite and hence can be used as the probability density. As a matter of fact, assuming that ψ is normalised according to

$$\int d^3x\psi^{\dagger}\psi = 1, \tag{2.37}$$

we can interpret $\psi^{\dagger}\psi d^3x$ as the probability of finding the Dirac particle in an element d^3x of volume in the xyz space at the instant t. **j** is then the vector for the probability current density specifying the mode of the probability flow.

2.3.5 Spin Angular Momentum

It can be seen that the orbital angular momentum operator does not commute with the Dirac Hamiltonian (2.24) for a particle in a central potential. So the orbital angular momentum is not a constant of motion. Let us now consider the operator

$$\mathbf{S} = \frac{1}{2}\mathbf{\Sigma}, \tag{2.38}$$

with

$$\mathbf{\Sigma} = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix} \tag{2.39}$$

The cartesian components of **S** will satisfy the commutation relations

$$[S_x, S_y] = iS_z, \qquad [S_y, S_z] = iS_x, \qquad [S_z, S_x] = iS_y. \tag{2.40}$$

So **S** can be taken as the operator for spin angular momentum as the eigenvalues of S^2 are $s(s + 1)$, where $s = 1/2$. We can now define a total angular momentum operator **J** as the sum of the orbital and spin angular momentum operators

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

Each component of **J** happens to commute with H and so is conserved. Thus, in Dirac's theory, simultaneous eigenstate of the Hamiltonian and of the operators J^2, J_z can be found, with eigenvalues E, $j(j + 1)$, m respectively. We, therefore, see that spin is a natural consequence in Dirac Theory.

2.3.6 Plane Wave Solutions of Dirac Equation

To solve the free-particle Dirac equation (2.18), we describe the electron with momentum

p and energy E (the momentum operator commutes with the Hamiltonian) by the plane wave

$$\psi(x) = u(p)e^{-ip\cdot x}. \tag{2.41}$$

The Dirac equation for a four-component particle spinor $u(p)$ then is

$$(\gamma \cdot p - m)u(p) = 0. \tag{2.41a}$$

It represents a set of four homogeneous linear equations and has four independent solutions. The condition for the non-trivial solutions to exist is

$$\det\,(\gamma \cdot p - m) = -\{E^2 - \mathbf{p}^2 - m^2\}^2 = 0. \tag{2.42}$$

For a given value of the momentum, therefore, the energy of the particle can have two values

$$E_\pm = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

Among the four solutions of the free particle Dirac equation, two correspond to the positive energy (particle) and two to the negative energy (corresponding to anti-particle). The two solutions of positive (or negative) energy are distinguished as the two spin states (up and down) due to the spin 1/2 of the particle. Taking the energy to be positive, the general solution can be written as

$$\psi = [u^1(p) + u^2(p)]e^{-ip\cdot x} + [v^1(p) + v^2(p)]e^{ip\cdot x}, \tag{2.43}$$

where

$$v^1(p) = u^3(-p), \qquad v^2(p) = u^4(-p). \tag{2.44}$$

For an antiparticle spinor $v(p)$, the Dirac equation is

$$(\gamma \cdot p + m)v(p) = 0, \tag{2.45}$$

whereas the equations for the adjoint spinors, $\bar{u} = u^\dagger\gamma^0$ and $\bar{v}$, are

$$\bar{u}(p)(\gamma \cdot p - m) = 0, \tag{2.46}$$

$$\bar{v}(p)(\gamma \cdot p + m) = 0. \tag{2.47}$$

With the form of the spinor u as

$$u(p) = \begin{pmatrix} \chi \\ \phi \end{pmatrix},$$

(2.41a) becomes

$$\left.\begin{aligned} (E - m)\chi - \boldsymbol{\sigma}\cdot\mathbf{p}\phi &= 0, \\ \boldsymbol{\sigma}\cdot\mathbf{p}\,\chi - (E + m)\phi &= 0. \end{aligned}\right\} \tag{2.48}$$

The normalized free particle solutions are

$$u(p) = \sqrt{(E + m)}\begin{pmatrix} \chi \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E + m}\chi \end{pmatrix}, \tag{2.49}$$

$$v(p) = \sqrt{(E+m)}\begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\phi \\ \phi \end{pmatrix}. \tag{2.50}$$

where χ and ϕ are two-component Pauli spinors. Physical states with given momentum **p** are taken to be normalized according to

$$\langle \mathbf{p}, \alpha | \mathbf{p}', \alpha' \rangle = 2E(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\alpha\alpha'}, \tag{2.51}$$

where α, α' represent collectively all other quantum numbers required to identify the states. The above normalization of states implies

$$\bar{u}(p)u(p) = 2m$$

which gives

$$u^\dagger(p)u(p) = 2E,$$

and

$$\bar{v}(p)v(p) = -2m,$$

yielding

$$v^\dagger(p)v(p) = -2E.$$

The sum over the polarization then gives

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m. \tag{2.52}$$

The covariant normalization of states implies, for the completeness, the integral

$$\sum_p |p\rangle\langle p| = \int \frac{d^3p}{2E(2\pi)^3} = \frac{1}{2\pi^3}\int d^4p\,\delta(p^2-m^2)\,\theta(p_0). \tag{2.53}$$

The matrix γ^5 is defined as

$$\gamma^5 \equiv \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.54}$$

γ^5 is Hermitian. We define also the antisymmetric $\sigma_{\mu\nu}$ by

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{2.55}$$

and the 4×4 $\boldsymbol{\Sigma}$ matrices defined by (2.39) can also be written as

$$\Sigma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = -\gamma^0\gamma^i\gamma^5, \tag{2.56}$$

which satisfy

$$[\Sigma^i, \gamma^5] = 0.$$

2.3.7 Dirac Bilinears

We give below Lorentz transformation laws of Dirac bilinears of the form

$$\bar{\psi}(x)\Gamma^A\psi(x). \tag{2.57}$$

Γ^A are the following sixteen 4×4 matrices which are linearly independent and form a complete set of 4×4 matrices in the sense that any 4×4 matrix can be written as a linear combination of these 16 matrices (see Table 2.1). These sixteen 4×4 matrices are obtained by multiplying two or more of the six matrices I, γ^0, γ^1, γ^2, γ^3, and γ^5. $\bar{\psi}(x)\psi(x)$ transforms like a scalar (S), whereas $\bar{\psi}(x)\gamma^\mu\psi(x)$ like a four vector (V). Slightly more subtle, $\bar{\psi}(x)\gamma^5\psi(x)$ transforms as a Lorentz scalar as well, but is odd under parity transformation, so it is called a pseudoscalar (P). And $\bar{\psi}(x)\gamma^5\gamma^\mu\psi(x)$ also transforms like a four vector but is even under the inversion of the coordinates (parity), so it is called an axial vector (A). $\bar{\psi}\sigma_{\mu\nu}\psi$ is a tensor. These properties are of use in nuclear and particle physics, particularly in the treatment of form factors in electromagnetic and weak interactions.

Table 2.1 Complete Set of 4 × 4 Matrices

Γ^A	*Number of matrices*	*Transformation law*
I (unit matrix)	1	S
γ^μ	4	V
γ^5	1	P
$\gamma^\mu\gamma^5$	4	A
$\sigma^{\mu\nu}$	6	T

2.3.8 Dirac Equation for a Massless Particle

For $m = 0$, the Dirac equation becomes

$$\gamma \cdot p u(p) = 0, \tag{2.58}$$

giving

$$\left.\begin{aligned} (\gamma^0 E - \boldsymbol{\gamma} \cdot \mathbf{p})u(p) &= 0, \\ (E - \gamma^0\boldsymbol{\gamma} \cdot \mathbf{p})u = (E + \gamma^5\boldsymbol{\Sigma} \cdot \mathbf{p})u &= 0, \\ \boldsymbol{\Sigma} \cdot \mathbf{p}\gamma^5 u &= Eu \end{aligned}\right\} \tag{2.59}$$

because $\gamma^5\Sigma^i = \gamma^0\gamma^i = \Sigma^i\gamma^5$, and multiplication on both sides by γ^5 yields

$$\boldsymbol{\Sigma} \cdot \mathbf{p}u = E\gamma^5 u$$

so that

$$\boldsymbol{\Sigma} \cdot \mathbf{p}(1 - \gamma^5)u = -E(1 - \gamma^5)u, \tag{2.60}$$

$$\boldsymbol{\Sigma} \cdot \mathbf{p}(1 + \gamma^5)u = +E(1 + \gamma^5)u. \tag{2.61}$$

Chirality

Chirality is the eigenvalue of γ_5, with $\gamma_5 = +1$ corresponding to right-handedness, and $\gamma_5 = -1$ to left-handedness:

$$\gamma_5\psi_L = -\,\psi_L, \qquad \bar{\psi}_R\gamma_5 = -\,\bar{\psi}_R,$$

$$\gamma_5\psi_R = \psi_R, \qquad \bar{\psi}_L\gamma_5 = \bar{\psi}_L,$$

where ψ_R, ψ_L are respectively the right-handed and left-handed Dirac spinors with only two independent components. These are obtained from a 4-component Dirac spinor ψ by the following projections:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi, \qquad \bar{\psi}_R = \bar{\psi}_R\,\frac{1}{2}(1-\gamma_5),$$

$$\psi_R = \frac{1}{2}(1+\gamma_5)\psi, \qquad \bar{\psi}_L = \bar{\psi}_L\,\frac{1}{2}(1+\gamma_5),$$

It is important to note that

$$\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R,$$

$$\bar{\psi}\gamma_\mu\psi = \bar{\psi}_R\gamma_\mu\psi_R + \bar{\psi}_L\gamma_\mu\psi_L.$$

2.4 Introduction to Quantum Field Theory

2.4.1 Introduction

In classical physics, classical mechanics describes a discrete system, a single particle or a system of finite number of particles. A continuous system with an infinite number of degrees of freedom is described by classical field theory. In quantum mechanics, the dynamical variables like the coordinates of a particle become operators. In field theory, the field plays a role similar to a coordinate. It is natural, therefore, to ask what would happen if a field becomes an operator. Not unrelated to this question, is a problem associated with the interpretation of the relativistic quantum mechanics. Relativistic quantum mechanics gives a description of the dynamics of a relativistic particle. But the interpretation of the negative energy solutions of relativistic quantum mechanical equations necessitates the existence of an infinite number of particles filling all the negative energy states. The relativistic quantum mechanics, therefore, has to be a theory of infinitely many particles. We shall see that a quantized field describes a system of infinite number of identical particles. Quantization of a field, therefore, not only satisfies a natural curiosity of finding out what quantization of a field would lead to, but it also removes the essential difficulty in the understanding of relativistic quantum mechanics. The construction of this many particle theory is called field quantization and the theory is known as quantum field theory.

2.4.2 Classical Field Theory

The concept of field was introduced in classical physics to explain the action at a distance. A classical field reveals itself by producing a force on a material object that happens to be in the space occupied by the field. The examples of classical fields are the electric and magnetic fields. The laws of behaviour of electric and magnetic fields were formulated by J.C. Maxwell. He found that in order to describe completely the state of the field in a given region of space, it is necessary to specify the strength and direction of both the electric and magnetic fields at every point of the region. This is the characteristic mathematical property of a classical field. It is something which exists throughout a volume of space and which is described by a set of numbers denoting the field strength and direction at each point in space. Maxwell was the first to realize that electric and magnetic fields could exist not only near charges but also in vacuum. The role of a field in a continuum is similar to the role of a coordinate for a particle or a system of particles. We, therefore, develop the Lagrangian and Hamiltonian formalism for a field on lines similar to the particle dynamics.

2.4.3 Discrete and Continuous Systems

The dynamical behaviour of a particle can be described by the Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.62}$$

This equation is obtained from Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0. \tag{2.63}$$

The Lagrangian L is given, in most cases, by

$$L = T - V, \tag{2.64}$$

where T is the kinetic energy and V the potential energy of the particle. The variation in (2.63) is to be taken over an arbitrary path $q_i(t)$ such that δq_i vanish at t_1 and t_2. Using $\delta_{\dot{q}} = (d/dt)\delta q$ and

$$\delta A = \int \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right] dt, \tag{2.65}$$

and integrating the second term by parts we obtain (2.62). The Hamiltonian of the system is defined by

$$H = \sum_i p_i \dot{q}_i - L, \tag{2.66}$$

where the canonical conjugate momenta p_i are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.67}$$

It is easy to generalize the Lagrangian formalism to $3n$ degrees of freedom (a system of n particles). We can also go from a discrete mechanical system to a continuous system where the number of degrees of freedom become infinite. For a classical field $\phi(\mathrm{x}, t)$ describing a continuous system, we divide all space into small cells and convert the continuous coordinate into discrete one. The analogue of $q_n(t)$ is then just the field at the centre of nth cell.

$$q_n(t) \to \phi(\mathrm{x}, t). \tag{2.68}$$

Classical field theory is the limit $n \to \infty$. We have to set up a variational principle for $\phi(\mathbf{x}, t)$. Since we want a relativistic theory, we want space and time to enter on equal footing. We, therefore, assume

$$L = \int d^3x\, \mathcal{L}(\mathbf{x}, t), \tag{2.69}$$

so that

$$A = \int d^4x\, \mathcal{L}(\mathbf{x}, t). \tag{2.70}$$

$\mathcal{L}$ is called the Lagrangian density. By requiring $\delta A = 0$, we derive

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \phi)} = 0 \tag{2.71}$$

which, in the covariant notation, may be written as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.72}$$

Define conjugate momentum $\pi(x)$ to the coordinate $\phi(x)$ by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{2.73}$$

x now represents the four coordinate $(\mathbf{x}, t)$. The Hamiltonian density is then given by

$$\mathcal{H}(x) = \pi(x)\dot{\phi}(x) - \mathcal{L}. \tag{2.74}$$

and the Hamiltonian is

$$H = \int d^3x \mathcal{H}. \tag{2.75}$$

The Hamilton equations of motion

$$\frac{\delta \mathcal{H}}{\delta \pi} = \dot{\phi}(x), \qquad \frac{\delta \mathcal{H}}{\delta \phi} = -\dot{\pi}(x), \tag{2.76}$$

then give us again the equation of motion of the field.

Choosing the Lagrangian

The Lagrangian density is a function only of the dynamical variables, i.e. the components of the field function $\phi(x)$ and their derivatives. Let us briefly list the main requirements imposed on the Lagrangian density in field theory:

1. The Lagrangian density must be Lorentz invariant.

2. The Lagrangian density for a free field must be bilinear (or quadratic) in the field variables. This condition ensures that the field equations are linear in field variables. If we wish to allow nonlinear field equations, this condition can be relaxed.

3. The Lagrangian density should not contain derivatives of higher order than one.

4. The Lagrangian density must be local. All the field variables in the Lagrangian density must be taken at the same space-time point.

To illustrate these rules, let us consider the simplest example of the non-interacting field theory, that of a real scalar field, $\phi(x)$, which describes spinless particles. Imposing the general restrictions, we can write the most general form of the free Lagrangian density as

$$\mathcal{L} = a\partial_\mu\phi\partial^\mu\phi + b\phi^2. \tag{2.77}$$

In order to obtain the Klein-Gordan equation, substituting (2.77) into the Euler-Lagrange (EL) equation, we get

$$b/a = -m^2. \tag{2.78}$$

For convenience we choose $a = 1/2$. So

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{2.79}$$

2.4.4 Field Quantization

The essential features of a quantum field theory are:

1. The field functions are linear operators instead of being c-numbers as in the classical case. This is analogous to the first quantization where the dynamical variables become operators.

2. These field operators obey certain commutation relations. These relations are the quantum conditions and the establishment of these relations is called the second quantization or field quantization.

3. The field operators are not observable themselves but the operators corresponding to the observable quantities are expressible in terms of the field operators.

4. The field operators and so the operators corresponding to physical observables act on state vectors (of a Hilbert space) representing an assembly of particles and cause the quantum states of that assembly to change.

Thus quantum field theories are intrinsically many particle theories.

2.5 A Real Scalar (KG) Field

The simplest field is a real scalar field $\phi(x)$. The simplest nontrivial Lagrangian density that can be constructed from such a field and its derivative is given in (2.79), where m is a constant. From the EL equation (2.72), the equation of motion is the Klein-Gordon equation. Separating the space and time derivatives, the Lagrangian density is expressed as in (2.79). Then the canonical conjugate field is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}.$$

The Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}[\pi^2 + \nabla\phi \cdot \nabla\phi + m^2\phi^2], \tag{2.80}$$

and so the Hamiltonian is

$$H = \int d^3x\mathcal{H} \tag{2.81}$$

The Hamilton equations of motion

$$\frac{\delta\mathcal{H}}{\delta\pi} = \dot{\phi}(x), \qquad \frac{\delta\mathcal{H}}{\delta\phi} = -\dot{\pi}(x), \tag{2.82}$$

then give us again the equation of motion of the field. We can write

$$H = \int \mathcal{H}d^3x = \int d^3x\, \frac{1}{2}[\pi^2 + \nabla\phi \cdot \nabla\phi + m^2\phi^2]. \tag{2.83}$$

Partial integration of the second term in the integrand yields

$$H = \int d^3x\, \frac{1}{2}[\pi^2 - \phi\nabla^2\phi + m^2\phi^2].$$

The general solution of KG equation at any time can be written as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\,k^0}}\,[a(\mathrm{k})e^{-ik\cdot x} + a^*(\mathrm{k})e^{ik\cdot x}] = \phi^{(+)}(x) + \phi^{(-)}(x). \tag{2.84}$$

So

$$\pi(x) = \dot{\phi} = (2\pi)^{3/2}\int \frac{d^3k(-ik^0)}{\sqrt{2k^0}}\,[a(\mathrm{k})e^{-ik\cdot x} - a^*(\mathrm{k})e^{ik\cdot x}]. \tag{2.85}$$

That the complex conjugate plane waves have complex conjugate coefficients is dictated by the reality of $\phi(x)$.

2.5.1 Application of Klein-Gordon Equation (Real field)

Yukawa potential

Let us now introduce an interaction of ϕ with a source. This can easily be incorporated into the Lagrangian (or Hamiltonian) formalism by adding the term

$$\mathcal{L}_{\text{int}} = -\rho\phi(x) \tag{2.86}$$

to $\mathcal{L}_{\text{free}}$ of (2.79). ρ is the source density, which, in general, is a function of space time coordinates. The total Lagrangian then is

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,m^2\phi^2 - \rho\phi(x). \tag{2.87}$$

The field equation now becomes

$$(\Box + m^2)\phi(x) = \rho(x). \tag{2.88}$$

Let us assume the source to be a point source at the origin independent of time. Let us consider a static solution. The equation becomes

$$(\nabla^2 - m^2)\phi(\mathbf{x}) = G\delta(\mathbf{x}), \tag{2.89}$$

where G is a constant which characterizes the strength of the coupling of the field with the source. G is analogous to e in electrodynamics. To solve (2.89), we use the Fourier transform method. Let us define

$$\tilde{\phi}(k) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(x) \tag{2.90}$$

so that

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\phi}(k). \tag{2.91}$$

Substitution of (2.91) into (2.89) immediately gives

$$\tilde{\phi}(k) = -\frac{G}{(2\pi)^{3/2}} \frac{1}{(|\mathbf{k}|^2 + m^2)}. \tag{2.92}$$

Therefore,

$$\phi(x) = -\frac{G}{(2\pi)^{3/2}} \int d^3 k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{(|\mathbf{k}|^2 + m^2)}. \tag{2.93}$$

After the integration has been performed, we obtain

$$\phi(x) = -\frac{Ge^{-mr}}{4\pi r}. \tag{2.94}$$

Yukawa identified ϕ as a meson field, with nucleon as source, just as an electrically charged particle is the source of the electromagnetic field. And just as the effects of the electromagnetic field are transmitted by photons, the effect of the meson field should be transmitted by particles called mesons. According to the Yukawa's theory of nuclear forces then, the nucleon (proton or neutron) is a source of a force field, called the meson field. Assuming that the static meson field due to a nucleon at the origin satisfied (2.89), the strength of the field at a point $\mathbf{x}$ due to nucleon at $\mathbf{x}_1$ is given by

$$\phi(x) = -\frac{Ge^{-m|\mathbf{x}-\mathbf{x}_1|}}{4\pi|\mathbf{x}-\mathbf{x}_1|}. \tag{2.95}$$

Let us now extend this to see how one nucleon would interact with another by generating its meson field. Since the interaction Hamiltonian density in this case is $-\mathcal{L}_{\text{int}}$ the total interaction Hamiltonian between two nucleons, the second nucleon described by the density $\rho(\mathbf{x}) = G\delta(\mathbf{x} - \mathbf{x}_2)$ is

$$H_{\text{int}} = \int \mathcal{H}_{\text{int}}\, d^3x = \int d^3x \rho(x)\phi(x). \tag{2.96}$$

The interaction energy between two nucleons is then

$$H_{\text{int}} = -\frac{G^2}{4\pi}\frac{e^{-m|\mathbf{x}_2 - \mathbf{x}_1|}}{|\mathbf{x}_2 - \mathbf{x}_1|}. \tag{2.97}$$

This interaction is attractive and short ranged; it goes to zero very rapidly for $|\mathbf{x}_2 - \mathbf{x}_1| >> \frac{1}{m}$.

2.5.2 Quantization of a Real Scalar Field

We have discussed the classical field theory of a real scalar field. In the quantum field theory, we take ϕ and π to be Hermitian operators. $\phi = \phi^\dagger$ and $\pi = \pi^\dagger$. The canonical commutation rules assumed are:

$$[\phi(x), \pi(x')]_{t'=t} = i\delta(\mathbf{x} - \mathbf{x}'), \qquad [\phi(x), \phi(x')]_{t'=t} = [\pi(x), \pi(x')]_{t'=t} = 0. \tag{2.98}$$

We can check explicitly that with the Hamiltonian, the Heisenberg equations of motion

$$i\dot{\phi} = [\phi, H], \qquad i\dot{\pi} = [\pi, H] \tag{2.99}$$

are identical in form with Hamilton's equation. The solution can be again written in the form (2.84) except that $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are now operators; $a^\dagger(\mathbf{k})$, the Hermitian conjugate of $a(\mathbf{k})$ now appearing in place of a^*. Note that $a^\dagger(-\mathbf{k}) = a(\mathbf{k})$ is a consequence of the condition of hermiticity. The solution thus is

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k_0}}[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}] = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{2.100}$$

$$\pi(x) \equiv \dot{\phi} = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k(-ik^0)}{\sqrt{2k^0}}[a(\mathbf{k})e^{-ik\cdot x} - a^\dagger(\mathbf{k})e^{ik\cdot x}]. \tag{2.101}$$

For the purposes of quantization we choose $t = 0$ as the commutation relations and the Hamiltonian does not depend upon time. Then

$$\phi(\mathbf{x}, 0) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k_0}}[a(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}] \tag{2.102}$$

$$\pi(\mathbf{x}, 0) = \frac{(-ik^0)}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k^0}}[a(\mathbf{k})e^{+i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{2.103}$$

The Fourier inversion of (2.102) and (2.103) gives

$$a(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3x}{\sqrt{2k^0}} e^{-i\mathbf{k}\cdot\mathbf{x}}[k^0\phi(x) + i\pi(x)], \tag{2.104}$$

$$a^{\dagger}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3x}{\sqrt{2}\,k^0} e^{+i\mathbf{k}\cdot\mathbf{x}} \left[k^0\phi(x) - i\pi(x)\right]. \tag{2.105}$$

Since $\phi(x)$ and $\pi(x)$ satisfy the commutation relations (2.98), it is found by direct calculation (taking $t = 0$ for simplicity because the commutation relations are valid at any time) that the operators $a(\mathbf{k})$ and $a^{\dagger}(\mathbf{k})$ must satisfy the relations

$$[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')] = \delta(\mathbf{k}, \mathbf{k}'), \qquad [a(\mathbf{k}), a(\mathbf{k}')] = [a^{\dagger}(\mathbf{k}), a^{\dagger}(\mathbf{k}')] = 0, \tag{2.106}$$

which are just the commutation relations for the creation and destruction operators of the number representation. We shall now define a number operator $N(\mathbf{k}) = a^{\dagger}(\mathbf{k})a(\mathbf{k})$ and a set of basic vectors $|n_1, n_2, n_3 \ldots n_N>$ (states of the Hilbert space called the Fock space) for which

$$N(\mathbf{k})\,|n_1 \ldots n_k \ldots\rangle = n_k\,|n_1 \ldots n_k \ldots\rangle, \tag{2.107}$$

where n_k is the number of quanta associated with momentum $\mathbf{k}$. That such a diagonal representation can be found for different $N(k)$ follows from the fact that

$$[N(\mathbf{k}), N(\mathbf{k}')] = 0. \tag{2.108}$$

The Hamiltonian can be expressed in terms of a and $a^{\dagger}$. At some fixed time (say $t = 0$), using (2.83), (2.84), (2.85), we find that

$$H = \sum_{\mathbf{k}} k_0 \left[a^{\dagger}(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right]. \tag{2.109}$$

The ground state energy corresponding to all $n_k = 0$ is not only nonvanishing but actually it is infinite. The divergence of the zero point energy is a general feature of the quantum field theory. It is a matter of conceptual difficulty but it does not lead to any infinite consequences in the practical predictions of the theory. As a matter of fact, it is the easiest one to remove simply by substracting the infinite constant to cancell the $\sum_{\mathbf{k}} \frac{1}{2}\omega_k$. We choose the zero of the energy scale for any oscillator to be the ground state.

This may be formally implemented with the help of the normal product operator, denoted by : : which, embracing a product of creation and annihilation operators, is defined to rearrange the product so that all the annihilation operators (positive frequency parts) stand to the right of creation operators (negative frequency parts). Because of the commutation rules, the order of the creation operators among themselves or annihilation operators among themselves is immaterial. In addition, : : acting on a linear combination of product of creation and annihilation operators is assumed to give the same linear combination of normally ordered product. Thus, for example,

$$:aa^{\dagger}: = a^{\dagger}a,$$

$$:aa^{\dagger} + aa^{\dagger}a: = a^{\dagger}a + a^{\dagger}aa,$$

$$:a^{\dagger}a + aa^{\dagger}: = 2a^{\dagger}a.$$

The redefined operator obeys the same commutation rules as the original operator. The same equations of motion remain valid. This is because they are quadratic function of the creation and destruction operators. The constant by which the redefined operator and the original operator differ are infinite but nevertheless c-numbers. Moreover, the vacuum expectation value of a normally ordered product is zero. We redefine

$$H = \sum_{\mathbf{k}} \frac{1}{2} : a^{\dagger}a + aa^{\dagger} : k_0 = \sum_{\mathbf{k}} n_k k_0. \tag{2.110}$$

The second quantized Klein-Gordon field, thus, gives a description of an assembly of infinitely many non-interacting particles—all identical, all spinless, each satisfying the equation with the same mass. The particles are not only spinless but also uncharged. That the associated quanta are uncharged follows from the fact that the four vector current density corresponding to Hermitian field is zero.

Vacuum state

We define the vacuum as the lowest energy state of the system. It has zero momentum and denote it by $|\text{vac}\rangle \equiv |0\rangle \equiv |\Phi_0\rangle$. In most quantum field theories, the vacuum is used only to enable us to perform mathematical construct of a Hilbert space. From the vacuum state, we build the one-particle state, then the two-particle states, and so on. Hopefully, the resulting Hilbert space will eventually resemble our universe. Different vacuum state means different Hilbert space and therefore different universe. Accordingly, we may say that the operator $a^{\dagger}(\mathbf{k})$ creates a particle of momentum **k** and the operator $a(\mathbf{k})$ destroys (or annihilates) a particle of momentum **k**.

Table 2.2 States of Hilbert Space of a Real Scalar Field

State	*Energy*	*Momentum*	*Description*	*Particle description*
$\|\text{vac}\rangle$	0	0	All oscillations in ground state	No particle
$\|\mathbf{k}\rangle \equiv a^{\dagger}(\mathbf{k})\|0\rangle$	$\omega_k \equiv k_0$	$\mathbf{k}$	Oscillations **k** in first excited state, all others in ground state	State of one particle of momentum **k**, mass m
$\|\mathbf{k}_1, \mathbf{k}_2\rangle$ symmetry	$\omega_1 + \omega_2$	$\mathbf{k}_1 + \mathbf{k}_2$	Oscillations $\mathbf{k}_1$ and $\mathbf{k}_2$ in first excited state, all others in ground state	State of two particles of momentum $\mathbf{k}_1$ and $\mathbf{k}_2$

The particles described by this theory are:

(i) *Spinless:* This is obvious since one particle states are completely specified by their momenta.

(ii) *Bosons:* From $[a^{\dagger}(\mathbf{k}_1), a^{\dagger}(\mathbf{k}_2)] = 0$, it follows that

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = |\mathbf{k}_2, \mathbf{k}_1\rangle \tag{2.111}$$

(iii) *Non-interacting:* This is because the number of the particles and their individual momenta are time independent.

Application of the concept of second quantization

The pion-nucleon interaction. To illustrate the concepts we have learnt so far, we wish to consider an example of the interaction of neutral pions with nucleons. According to the Yukawa theory of nuclear interactions, the nucleon is a source of the pion field. In analogy with electrostatics, we can write the interaction Hamiltonian density, as

$$\mathcal{H}_I = g\rho(x)\phi(x), \tag{2.112}$$

where g is the coupling constant and measures the strength of the interaction. $\rho(x)$ is the nucleon density. The total Hamiltonian density is

$$\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{free}} + \mathcal{H}_I, \tag{2.113}$$

$$H_{\text{total}} = \int d^3x \mathcal{H}_{\text{total}}. \tag{2.114}$$

In the quantized framework, $\phi(x)$ is an operator satisfying quantum conditions. The Hamiltonian is also an operator then operating on the Fock-space.

We shall treat the nucleons as point particles of large mass so that they can be taken as fixed at $\mathrm{x}_1, \mathrm{x}_2 \ldots$. The nucleon density can then be written as

$$\rho(\mathrm{x}) = \sum_n \delta(\mathrm{x} - \mathrm{x}_n). \tag{2.115}$$

The interaction Hamiltonian then becomes

$$\begin{aligned} H_I &= \int d^3x\, \mathcal{H}_I \\ &= \sum_n \int d^3x \delta(\mathrm{x} - \mathrm{x}_n) \frac{g}{\sqrt{V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{2k^0}} [a(\mathrm{k}) e^{-i\mathrm{k}\cdot\mathrm{x}} + a^\dagger(\mathrm{x}) e^{i\mathrm{k}\cdot\mathrm{x}}] \\ &= \frac{g}{\sqrt{V}} \sum_{\mathrm{k}} \sum_n \frac{1}{\sqrt{2k_0}} [a(\mathrm{k}) e^{-i\mathrm{k}\cdot\mathrm{x}_n} + a^\dagger(\mathrm{k}) e^{i\mathrm{k}\cdot\mathrm{x}_n}]. \end{aligned} \tag{2.116}$$

This Hamiltonian is linear in destruction and creation operators. Each term of the first or second type will annihilate or create a meson of momentum k. Hence the only non-zero matrix elements are

$$\langle \ldots n_k + 1, \ldots | H_I | \ldots n_k \ldots \rangle = \frac{g}{\sqrt{V}} \sqrt{\frac{(n_k + 1)}{2\omega_k}} \sum_n e^{i\mathrm{k}\cdot\mathrm{x}_n}, \tag{2.117}$$

the other nonzero matrix element $\langle \ldots n_k - 1 \ldots | H_I | \ldots n_k \ldots \rangle$ being related to the above matrix element through complex conjugation. The occupation number n_k refers to the momentum state k, all other occupation numbers remaining unchanged in these transitions; that is, the only allowed transitions are those changing the occupation number of a single state by one. Nucleons are treated classically. They merely serve as sources and sinks of the meson field, which is quantized leading to the possibility of destruction and creation of mesons. The number of nucleons and their positions are fixed, so that the state vectors representing the system will be a function of the meson occupation numbers only. For example, the vacuum state will now be the state with only the N nucleons present but no mesons.

As first example, let us consider the nuclear force problem, i.e. we shall calculate the change in energy ΔE, between two nucleons at $\mathbf{x}_1$ and $\mathbf{x}_2$ due to their interaction through the meson field. We shall use the time independent perturbation theory and choose $t = 0$. The unperturbed wavefunction is now the vacuum state Φ_0, consisting of two nucleons but no mesons. In the first order,

$$\Delta E = \langle \Phi_0 \,|H_I|\, \Phi_0 \rangle. \tag{2.118}$$

This is zero. In the second order perturbation approximation,

$$\Delta E = \sum_{\mathbf{k}} \frac{\langle \Phi_0 | H_I | \Phi(\mathbf{k}) \rangle \langle \Phi(\mathbf{k}) \,| H_I |\, \Phi_0 \rangle}{-\,\omega_k}. \tag{2.119}$$

Here $\Phi(\mathbf{x})$ is the state with one meson of momentum $\mathbf{k}$.

$$\Delta E = -\frac{g^2}{V} \sum_{\mathbf{k}} \frac{1}{2\omega_k^2} \sum_{n=1}^{2} \sum_{m=1}^{2} e^{i\mathbf{k}\cdot(\mathbf{x}_n - \mathbf{x}_m)}. \tag{2.120}$$

($e^{-i\mathbf{k}\cdot\mathbf{x}_m}$ corresponding to creation, and $e^{i\mathbf{k}\cdot\mathbf{x}_n}$ corresponding to destruction). Summation on m and n gives rise to four terms and they can be represented diagrammatically (Fig. 2.1). Two of the diagrams corresponding to $m = n = 1$, $m = n = 2$ contribute to the self energy of the nucleon and the other two to the nucleon-nucleon interaction which we are interested in. For the nucleon-nucleon interaction, taking the continuum spectrum, we obtain

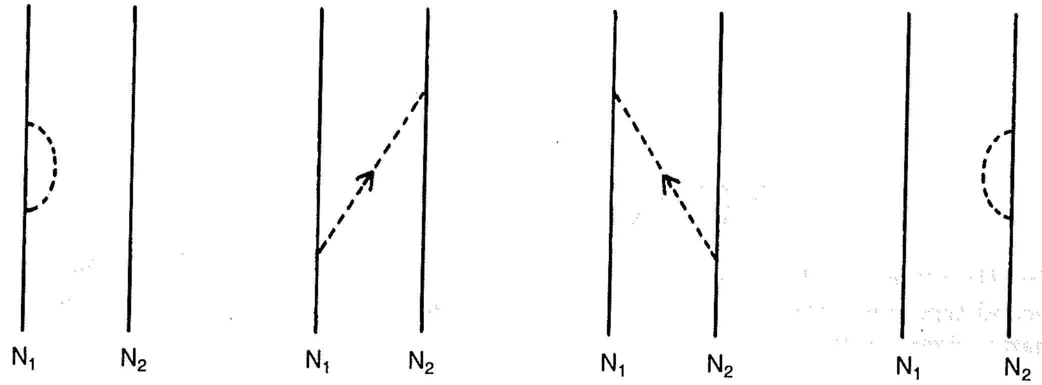

Fig. 2.1 Nucleon-nucleon interaction through an exchange of a pion.

$$\Delta E = -\frac{g^2}{(2\pi)^3} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m^2}, \qquad \omega_k^2 = \mathbf{k}^2 + m^2 = (k_0)^2.$$

where $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$. Three-dimension integration gives

$$\Delta E = -\frac{g^2}{4\pi r} e^{-mr}, \tag{2.121}$$

($r = |\mathbf{x}|$ is the internucleon distance). Equation (2.121) is nothing but the Yukawa potential.

2.5.3 Generalization for *N*-real Fields

The formalism can easily be generalized to the case of N real fields ϕ_i ($i = 1, 2, 3, \ldots, N$)

$$\mathcal{L} = \sum_i \mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{2.122}$$

Then N-field equations are

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial \phi_i/\partial x^\mu)}\right] = 0 \tag{2.123}$$

2.5.4 Complex Scalar Field

Let us consider two real independent fields of identical mass m. The free Lagrangian density is merely the sum of the free Lagrangian densities of the individual fields.

$$\begin{aligned}\mathcal{L}_{\text{free}}(\phi_1, \phi_2) &= \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2)\\ &= \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 \phi_2^2.\end{aligned} \tag{2.124}$$

We construct complex fields ϕ and ϕ^* given by

$$\phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}, \qquad \phi^* = \frac{(\phi_1 - i\phi_2)}{\sqrt{2}}. \tag{2.125}$$

Then

$$\phi_1 = \frac{1}{\sqrt{2}}(\phi + \phi^*), \qquad \phi_2 = \frac{1}{\sqrt{2}i}(\phi - \phi^*). \tag{2.126}$$

In terms of the complex fields as defined in (2.125), the Lagrangian density is

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \phi^*, \partial_\mu \phi^*) = \partial_\mu \phi^*_{(x)} \partial^\mu \phi_{(x)} - m^2 \phi^* \phi. \tag{2.127}$$

Substitution of (2.127) in the field equations yields

$$(\Box + m^2)\phi = 0, \tag{2.128}$$

and its conjugate

$$(\Box + m^2)\phi^* = 0. \tag{2.129}$$

We obtain the field equations from the variational principle by treating ϕ and ϕ^* as two independent fields. Then

$$\left.\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} &= 0 \Rightarrow (\Box + m^2)\phi = 0,\\ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= 0 \Rightarrow (\Box + m^2)\phi^* = 0.\end{aligned}\right\} \tag{2.130}$$

Physical interpretation of a complex field

Let us consider a unitary transformation

$$\phi \to \phi' = e^{i\lambda e}\phi, \qquad \phi^* \to \phi^{*\prime} = e^{-i\lambda e}\phi^*. \tag{2.131}$$

The free Lagrangian is clearly invariant under this transformation. Then from the discussion of KG equation in Section 2.3.2 or from Noether's theorem (discussed in Appendix C), we have

$$\frac{\partial f_\mu}{\partial x_\mu} = 0, \tag{2.132}$$

with

$$f_\mu = \frac{\partial \mathcal{L}}{\partial^\mu \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial^\mu \phi^*}\,\delta\phi^* = ie\left(\frac{\partial \phi^*}{\partial x^\mu}\,\phi - \phi^* \frac{\partial \phi}{\partial x^\mu}\right). \tag{2.133}$$

This means that there exists a conserved current f_μ associated with the complex field ϕ. Under the substitution $\phi \leftrightarrow \phi^*$, f_μ changes sign. This suggests that f_μ is to be interpreted as the electric current density and if ϕ corresponds to a particle with charge e, then ϕ^* is a field corresponding to a particle with charge $-e$. This can also be seen in the presence of the four vector potential, A_μ. We can easily show that if ϕ is a solution to the KG equation in the presence of A_μ with charge e, then ϕ^* is a solution to the KG equation in the presence of the same A_μ but with charge $-e$.

It is a remarkable feature of the relativistic field theory that it can readily accommodate a pair of particles and antiparticles.

2.5.5 Charged Scalar Field

We now turn to the more general case where the field function $\phi(x)$ is complex or non-Hermitian. A real scalar field describes suitably the behaviour of neutral spinless particles as it does not allow a distinction between particles and antiparticles. For the description of the charged spinless particles, a non-Hermitian (complex) field must be introduced.

Second quantization (Canonical)

We can write the Fourier expansion of the fields as

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2k^0}} \left[a(\mathbf{k})e^{-ik\cdot x} + b^\dagger(\mathbf{k})e^{ik\cdot x}\right] \\ &= \phi^{(+)}(x) + \phi^{(-)}(x),\end{aligned} \tag{2.134}$$

$$\phi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2k^0}} \left[b(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right] \tag{2.135}$$

$$= \phi^{\dagger(+)}(x) + \phi^{\dagger(-)}(x). \tag{2.136}$$

The fields conjugate to $\phi(x)$ and $\phi^\dagger(x)$ are

$$\pi(x) = \dot{\phi}^\dagger, \qquad \pi^\dagger(x) = \dot{\phi}, \tag{2.137}$$

$\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ are the positive and negative frequency parts of $\phi(x)$. The quantum conditions for the complex field are

$$[a(\mathrm{k}), a^\dagger(\mathrm{k}')] = \delta(\mathrm{k}, \mathrm{k}'), \qquad [b(\mathrm{k}), b^\dagger(\mathrm{k}')] = \delta(\mathrm{k}, \mathrm{k}'). \tag{2.138}$$

$\delta(\mathrm{k}, \mathrm{k}')$ stands for $\delta_{\mathrm{k},\mathrm{k}'}$ for the discrete spectrum and for $\delta(\mathrm{k} - \mathrm{k}')$ for the continuum. All other commutation relations vanish. These relations are equivalent to

$$[\phi(\mathrm{x}, t), \pi(\mathrm{x}', t)] = i\delta(\mathrm{x} - \mathrm{x}'). \tag{2.139}$$

The Hamiltonian density operator is

$$\mathcal{H} = \pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi. \tag{2.140}$$

When (2.134), (2.135) and (2.138) are used in (2.140), one gets for the Hamiltonian operator (after subtracting the infinite zero-point energy)

$$H = \int d^3x\mathcal{H}(x) = \sum_{\mathrm{k}}[a^\dagger(\mathrm{k})a(\mathrm{k}) + b^\dagger(\mathrm{k})b(\mathrm{k})]\omega_k \tag{2.141}$$

$$= \sum_{\mathrm{k}} [N^{(+)}(\mathrm{k}) + N^{(-)}(\mathrm{k})]\omega_k, \tag{2.142}$$

where $N^{(+)}(\mathrm{k}) = a^\dagger(\mathrm{k})a(\mathrm{k})$ and $N^{(-)}(\mathrm{k}) = b^\dagger(\mathrm{k})b(\mathrm{k})$ are the number operators corresponding to the quanta associated with field ϕ and $\phi^\dagger$ for a given momentum. In order to clarify the difference between two types of particles, we turn to the charge operator. From the invariance of the Lagrangian density (2.127) under the phase transformation (2.131), follows the conservation of charge Q, which takes the form

$$Q = \int d^3x j^0, \tag{2.143}$$

where

$$j^\mu(x) = -iq\,[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)]. \tag{2.144}$$

So

$$Q = -iq\int d^3x[\dot{\phi}^\dagger(x)\phi(x) - \phi^\dagger(x)\dot{\phi}(x)]. \tag{2.145}$$

Expressed in terms of creation and destruction operators,

$$Q = q\sum_{\mathrm{k}} [a^\dagger(\mathrm{k})a(\mathrm{k}) - b^\dagger(\mathrm{k})b(\mathrm{k})] = q\sum_{\mathrm{k}} [N^{(+)}(\mathrm{k}) - N^{(-)}(\mathrm{k})]. \tag{2.146}$$

Q commutes with the Hamiltonian. It follows from (2.146) that one must associate charge $+q$ with particles (associated with ϕ) and $-q$ with antiparticles (associated with $\phi^\dagger$). ϕ and $\phi^\dagger$ can be used to describe the charged pions π^+ and π^-. The state vectors are

$$|n_1^+ \ldots n_k^+ \ldots; n_1^- \ldots n_k^- \ldots\rangle. \tag{2.147}$$

Physical significance of field operators

$$\phi^{(+)}(x) = \frac{1}{\sqrt{V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{2\omega_k}} a(\mathrm{k}) e^{-ik\cdot x} \text{ destroys particle of charge } + e,$$

$$\phi^{(-)}(x) = \frac{1}{\sqrt{V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{2\omega_k}} b^{\dagger}(\mathrm{k}) e^{-ik\cdot x} \text{ creates particle of charge } - e,$$

$$\phi^{\dagger(-)}(x) = \frac{1}{\sqrt{V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{2\omega_k}} a^{\dagger}(\mathrm{k}) e^{-ik\cdot x} \text{ creates particle of charge } + e,$$

$$\phi^{\dagger(+)}(x) = \frac{1}{\sqrt{V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{2\omega_k}} b(\mathrm{k}) e^{-ik\cdot x} \text{ destroys particle of charge } - e.$$

Thus $\phi(x)$ creates particles of charge $+e$ and destroys particle of charge $-e$; and $\phi^{+}(x)$ creates particles of charge $-e$ and destroys particle of charge $+e$.

2.5.6 Free Dirac Field

The Dirac equation for a free particle of rest mass m can be written as

$$i\gamma^{\mu} \frac{\partial \psi(x)}{\partial x^{\mu}} - m\psi(x) = 0, \tag{2.148}$$

where γ^{μ} are 4×4 Dirac matrices satisfying anticommutation relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} \equiv \{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}, \tag{2.149}$$

$\psi(x)$ is thus a spinor wave-function with four components. The adjoint field $\overline{\psi}(x)$ defined by

$$\overline{\psi}(x) = \psi^{\dagger}(x)\gamma^{0} \tag{2.150}$$

satisfies the adjoint equation

$$i \frac{\partial \overline{\psi}(x)}{\partial x^{\mu}} \gamma^{\mu} + m\overline{\psi}(x) = 0. \tag{2.151}$$

The Dirac equation (2.148) can be derived from the EL equation if we choose

$$\mathcal{L} = - \overline{\psi}(x)(- i\gamma^{\mu}\partial_{\mu} + m)\psi(x) \tag{2.152}$$

as Lagrange function. Because

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = - (- i\gamma^{\mu}\partial_{\mu} + m)\psi(x), \qquad \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\overline{\psi})} = 0, \tag{2.153}$$

The Euler-Lagrange equation

$$\Rightarrow (- i\gamma^{\mu}\partial_{\mu} + m)\psi(x) = 0. \tag{2.154}$$

One could perhaps think that this Lagrangian is identically zero, since

$$(-i\gamma^\mu\partial_\mu + m)\psi(x) = 0,$$

according to the Dirac equation. But $\psi(x)$ in (2.152) is not a solution of the Dirac equation. Only when used in the variational principle

$$\delta \int \mathcal{L}\, d^4x = 0,$$

the particular $\psi(x)$ that makes the action stationary is a solution of the Dirac equation. For this $\psi(x)$, it happens that $\mathcal{L}$ is zero. But that is irrelevant for the variational argument. The canonical conjugate momentum field is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger. \tag{2.155}$$

The Hamiltonian density becomes

$$\begin{aligned}\mathcal{H} &= \pi\dot{\psi} - \mathcal{L} = i\psi^\dagger\dot{\psi} + \bar{\psi}\ (-i\gamma^\mu\partial_\mu + m)\psi \\ &= \pi\dot{\psi} = i\psi^\dagger\dot{\psi}. \end{aligned}\tag{2.156}$$

The total Hamiltonian is then

$$H = \int d^3x\ \mathcal{H}$$

2.5.7 Quantization of a Free Dirac Field

We constructed the real and complex scalar field operators by first expressing the classical fields as a Fourier series, and then replacing the amplitude of each plane wave in the series by an operator. From these operators, the operators for the total energy, and the total charge were constructed. We do the same for the Dirac field. The Lagrangian density and the Hamiltonian density for the Dirac field have been given in a previous section. We expand $\psi(x)$ in terms of the complete set of plane waves

$$u_r(\mathbf{p})e^{-ip\cdot x}, \qquad v_r(\mathbf{p})e^{ip\cdot x} \qquad (r = 1, 2),$$

where u_r and v_r are Dirac spinors. We treat $\psi(x)$ as an operator, not as a state vector, and expand in plane waves

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p},r} \sqrt{\frac{m}{E_p}}\ [c_r(\mathbf{p})u_r(\mathbf{p})e^{-ip\cdot x} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ip\cdot x}] \tag{2.157}$$

for discrete spectrum, or

$$\begin{aligned}\psi(x) &= \frac{1}{(2\pi)^{3/2}} \sum_r \int d^3p \sqrt{\frac{m}{E_p}}\ [c_r(\mathbf{p})u_r(\mathbf{p})e^{-ip\cdot x} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ip\cdot x}] \\ &= \psi^{(+)}(x) + \psi^{(-)}(x),\end{aligned}\tag{2.158}$$

for the continuous spectrum and so the adjoint field $\bar{\psi} = \psi^\dagger \gamma^0$ has the expansion

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_r \int d^3p \sqrt{\frac{m}{E_p}} \, [d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip\cdot x} + c_r^\dagger(\mathbf{p}) \, \bar{u}_r(\mathbf{p}) e^{ip\cdot x}]$$

$$= \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x), \tag{2.159}$$

where

$$\bar{u}_r = u_r^\dagger \gamma^0, \qquad \bar{v} = v^\dagger \gamma^0. \tag{2.160}$$

The spinors satisfy the orthonormality relations

$$u_r^\dagger(p) u_s(p) = v_r^\dagger(p) v_s(p) = \frac{E_p}{m} \delta_{rs},$$

$$u_r^\dagger(p) v_s(-p) = 0.$$

Anticommutation rules

We now establish the rules of quantization. Since the Dirac equation describes spin 1/2 particles which obey the Pauli-exclusion principle and the Fermi-Dirac statistics, we shall use the anticommutation relations for the expansion coefficients.

$$\{c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')\} = \{d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')\} = \delta_{rs} \delta_{\mathbf{p},\mathbf{p}'}, \tag{2.161}$$

and all other anticommutators vanish. That is to say,

$$\left.\begin{aligned} \{c_r(\mathbf{p}), c_s(\mathbf{p}')\} &= \{d_r(\mathbf{p}), d_s(\mathbf{p}')\} = \{c_r^\dagger(\mathbf{p}), c_s^\dagger(\mathbf{p}')\} = 0, \\ \{d_r^\dagger(\mathbf{p}), d_s^\dagger(\mathbf{p}')\} &= \{c_r(\mathbf{p}), d_s(\mathbf{p}')\} = 0, \\ \{c_r^\dagger(\mathbf{p}), d_s(\mathbf{p}')\} &= \{c_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')\} = \{c_r^\dagger(\mathbf{p}), d_s^\dagger(\mathbf{p}')\} = 0. \end{aligned}\right\} \tag{2.162}$$

These relations are equivalent to

$$\{\pi(x), \psi(x')\}_{t=t'} = -i\delta(\mathbf{x} - \mathbf{x}'), \tag{2.163}$$

with all other anticommutators equal to zero. We introduce the particle number operators

$$N_r^{(+)}(\mathbf{p}) = c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}), \qquad N_r^{(-)}(\mathbf{p}) = d_r^\dagger(\mathbf{p}) d_r(\mathbf{p}). \tag{2.164}$$

The interpretation of $c_r, c_r^\dagger, N_r^{(+)}$ and of $d_r, d_r^\dagger, N_r^{(-)}$ as destruction, creation and number operators of the two kinds of particles follows from

$$[N_r, c_s] = -\delta_{rs} c_s, \qquad [N_r, c_s^\dagger] = \delta_{rs} c_s^\dagger, \tag{2.165}$$

which are easily verified from (2.161) and (2.162) using

$$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{2.166}$$

Taking the time derivative of (2.158), we have

$$i\frac{\partial\psi}{\partial x^0} = \frac{1}{(2\pi)^{3/2}} \sum_r \int d^3p \sqrt{\frac{m}{E}}\, [c_r(\mathbf{p})u_r(\mathbf{p})\, e^{-ip\cdot x} - d^\dagger(\mathbf{p})v_r(\mathbf{p})\, e^{ip\cdot x}](\mathbf{p}_0). \tag{2.167}$$

We may write

$$\psi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \sum_r \int d^3p \sqrt{\frac{m}{E}}\, [c_s^\dagger(\mathbf{p}')u_s^\dagger(\mathbf{p}')\, e^{ip'\cdot x} + d_s(\mathbf{p}')v_s^\dagger(\mathbf{p}')\, e^{-ip'\cdot x}]. \tag{2.168}$$

Substituting (2.167) and (2.169) in the expression for the Hamiltonian,

$$H = \int d^3x\mathcal{H} = \int d^3x\,\psi^\dagger i\frac{\partial\psi}{\partial x^0}, \tag{2.169}$$

and using the anticommutation relations (2.161) and (2.162), we obtain

$$H = \sum_{\mathbf{p},\, r} [c_r^\dagger(\mathbf{p})\,c_r(\mathbf{p}) + d_r^\dagger(\mathbf{p})\,d_r(\mathbf{p}) - 1]\,p_0. \tag{2.170}$$

Subtracting the zero-point energy or vacuum energy, $E_\infty = \Sigma_{\mathbf{p},\, s}\, p_0$, we get

$$H - E_\infty = H' = \sum_{\mathbf{p},\, s} [c_s^\dagger(\mathbf{p})\,c_s(\mathbf{p}) + d_s^\dagger(\mathbf{p})\,d_s(\mathbf{p})]\,p_0. \tag{2.171}$$

The Lagrangian density is invariant under the phase transformation

$$\psi(x) \to \psi'(x) = e^{i\lambda q}\,\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\lambda q}\,\bar{\psi}(x), \tag{2.172}$$

and

$$\delta\psi = i\lambda\psi, \qquad \delta\bar{\psi} = -i\lambda\psi \quad \text{for small } \lambda. \tag{2.173}$$

Hence Noether's theorem gives the conserved current

$$j^\mu = -q\bar{\psi}\gamma^\mu\psi, \tag{2.174}$$

and the conserved charge

$$Q = q\int d^3x\,\psi^\dagger(x)\,\psi(x). \tag{2.175}$$

$N^{(+)}(p)$ represents the number operator for electrons carrying charge $-e$, and $N^{(-)}(p)$ stands for the number operator for positron with a charge $+e$. The first term $\psi^{(+)}$ in (2.158) is an operator which annihilates electrons of momentum $\mathbf{p}$, energy $E = p_0$ in the spin state r, and the second term $\psi^{(-)}$ in (2.158) creates a positron of the same energy and momentum. (±) correspond to the sign of exponential. With this convention the positive frequency spinor operators correspond to destruction (annihilation) and the negative frequency operators to creation of a fermion. That is, electron annihilation is associated with

$$\psi^{(+)} = \frac{1}{(2\pi)^{3/2}} \int d^3p \sum_r \sqrt{\frac{m}{E_p}}\, c_r(\mathbf{p})\,u_r(\mathbf{p})\,e^{-ip\cdot x}, \tag{2.176}$$

positron annihilation with

$$\overline{\psi}^{(+)} = \frac{1}{(2\pi)^{3/2}} \int d^3p \sum_r \sqrt{\frac{m}{E_p}}\, d_r(\mathbf{p}) \overline{v}_r(\mathbf{p}) e^{-ip\cdot x}, \tag{2.177}$$

electron creation with

$$\overline{\psi}^{(-)} = \frac{1}{(2\pi)^{3/2}} \int d^3p \sum_r \sqrt{\frac{m}{E_p}}\, c_r^{\dagger}(\mathbf{p}) \overline{u}_r(\mathbf{p}) e^{ip\cdot x}, \tag{2.178}$$

and positron creation with

$$\psi^{(-)} = \frac{1}{(2\pi)^{3/2}} \int d^3p \sum_r \sqrt{\frac{m}{E_p}}\, d_r^{\dagger}(\mathbf{p}) v_r(\mathbf{p}) e^{ip\cdot x}. \tag{2.179}$$

So ψ either annihilates an electron or creates a positron, and $\overline{\psi}$ either creates an electron or destroys a positron. The state vectors are now $|0\rangle$, $|\mathbf{p}, r\rangle = c_r^{\dagger}(\mathbf{p})|0\rangle$, ...

2.5.8 Vector Mesons

In addition to particles of spin 0 and 1/2, particles of spin 1 also play a fundamental role in (particle) physics. We shall now discuss the vector field which is used to describe particles with spin 1 and then the electromagnetic field which describes photons. We require that each component of a vector field satisfies the Klein-Gordon equation

$$(\Box + m^2)A^{\mu}(x) = 0. \tag{2.180}$$

This will imply that the quanta of the field, $A^{\mu}(x)$ will have mass m. Because the field has four components, we need a subsidiary condition

$$\partial_{\mu}A^{\mu}(x) = 0, \tag{2.181}$$

to describe a field spin of whose quanta is 1, as the three remaining independent components correspond to the three possible values of the components of spin along z-axis. The imposition of the subsidiary condition is equivalent to the exclusion of a particle of spin zero. The Lagrangian for the vector field can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_{\nu}A^{\nu}, \tag{2.182}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{2.183}$$

The equations of motion immediately give us

$$\partial_{\nu}F^{\nu\mu} + m^2A^{\mu} = -(\Box + m^2)A^{\mu} + \partial^{\mu}\partial_{\nu}A^{\nu} = 0. \tag{2.184}$$

These equations are called Proca's equations. Differentiation of this equation gives us the subsidiary condition. Thus, Proca's equations are equivalent to the set of Klein-Gordon equations along with the subsidiary condition.

2.5.9 Radiation Field and Photons

The classical theory of radiation is based on basic laws of classical electromagnetism, the Maxwell's equations for the electromagnetic field. The two fundamental quantities describing the electromagnetic field are the electric and magnetic field strengths, **E** and **B**, which are both functions of space and time. They satisfy the Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.185}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{2.186}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.187}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}. \tag{2.188}$$

Potentials

The field equations can be reduced to simpler equations in terms of a vector potential **A** and a scalar potential ϕ instead of two vectors **E** and **B**. The second and third Maxwell's equations are used to write

$$\left.\begin{aligned} \mathbf{B} &\equiv \nabla \times \mathbf{A}, \\ \mathbf{E} &\equiv -\nabla\,\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \end{aligned}\right\} \tag{2.189}$$

The remaining equations are reduced to two inhomogeneous coupled second order differential equations:

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho, \tag{2.190}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j}. \tag{2.191}$$

The solution of Maxwell's equations is thus reduced to solving the coupled equations (2.190) and (2.191) for **A** and ϕ.

2.5.10 Gauge Transformations

We now employ a fundamental property of classical electrodynamics, gauge invariance, to decouple the equations. The potentials **A** and ϕ as defined above are not unique. The transformation

$$\mathbf{A} \to \mathbf{A}' \equiv \mathbf{A} + \nabla\chi, \tag{2.192}$$

where χ is an arbitrary scalar function, will leave **B** unchanged due to the identity

$$\nabla \times \nabla\chi \equiv 0.$$

If we simultaneously change ϕ by

$$\phi \to \phi' \equiv \phi - \frac{\partial \chi}{\partial t} \tag{2.193}$$

E too will remain unchanged. These gauge transforms (**A**′, ϕ') by construction yield the same **E** and **B** as the potentials (**A**, ϕ) before the transformation, so the transformations leave the Maxwell's equations invariant. We can now use this invariance to choose a family of potentials (**A**, ϕ) such that

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \tag{2.194}$$

Then the second order equations decouple and we get

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho, \tag{2.195}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}. \tag{2.196}$$

For free field case, therefore, we have

$$\Box A^\mu = 0, \tag{2.197}$$

with the subsidiary condition written as

$$\partial_\mu A^\mu = 0, \tag{2.198}$$

This is termed as the Lorentz condition and the choice of the gauge as the Lorentz gauge or more precisely as the Lorentz family of gauges. There are other gauges as well which are useful for certain purposes. The Coulomb gauge, also known as the radiation gauge or transverse gauge, is specified by the condition

$$\nabla \cdot \mathbf{A} = 0. \tag{2.199}$$

Note that the Lorentz condition is covariant whereas the Coulomb condition is not. However, relativistic invariance is not lost for the Coulomb gauge.

Fourier expansion

We can assume the plane wave solution of the form

$$A^\mu = N\varepsilon^\mu e^{-ik\cdot x},$$

with $k^2 = 0$ (i.e. $k_0^2 = \mathbf{k}^2$), where ε^μ is a polarization vector of the wave and N is the normalization factor. The gauge condition (2.198) reduces to the condition on ε^μ, namely,

$$k \cdot \varepsilon = 0.$$

ε^μ is arbitrary in the sense that $\varepsilon^\mu \to \varepsilon'^\mu = \varepsilon^\mu + \beta k^\mu$ still satisfies the condition $\varepsilon' \cdot k = 0$ because $k^2 = 0$, for the free field solution. This freedom has important consequence. Consider a solution with $k^\mu = (k^0, \mathrm{k})$ and the polarization vector $\varepsilon^\mu = (\varepsilon^0, \varepsilon)$ with $k \cdot \varepsilon = 0$. Gauge invariance implies that we can add multiple of k^μ to ε^μ and both the polarization vectors will be satisfactory. Choosing the time component to be zero, then the Lorentz condition gives

$$\mathbf{k} \cdot \boldsymbol{\varepsilon} = 0, \tag{2.200}$$

i.e. the Coulomb gauge condition. For a wave travelling in the positive z direction, the two independent polarization vectors can be chosen as

$$\boldsymbol{\varepsilon}_1 = (1, 0, 0), \qquad \boldsymbol{\varepsilon}_2 = (0, 1, 0).$$

These correspond to the linear polarization of the associated **E** and **B**. The vector potential is then

$$A_i^\mu = N(0, \boldsymbol{\varepsilon}_i)e^{-ik \cdot x}, \qquad i = 1, 2.$$

The linear combinations

$$\boldsymbol{\varepsilon}(\lambda = +1) = -\frac{1}{\sqrt{2}}(1, i, 0),$$

$$\boldsymbol{\varepsilon}(\lambda = -1) = \frac{1}{\sqrt{2}}(1, -i, 0),$$

corresponding to left (helicity –1) and right (helicity +1) circular polarizations are commonly used. The corresponding 4-vector satisfies the relation

$$\varepsilon^*(\lambda) \cdot \varepsilon(\lambda') = -\delta_{\lambda\lambda'}.$$

The expansion of A_μ is given by

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\mathbf{k}) \{a^{(\lambda)}(\mathbf{k})e^{-ik\cdot x} + a^{*(\lambda)}(\mathbf{k})e^{ik\cdot x}\}$$

where $\varepsilon_\mu^\lambda(\mathbf{k})$ form a set of four mutually orthogonal unit polarization vectors.

2.5.11 Quantization of Electromagnetic Field

The expansion of the operator A_μ is given by

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\mathbf{k}) \{a^{(\lambda)}(\mathbf{k})e^{-ik\cdot x} + a^{\dagger(\lambda)}(\mathbf{k})e^{ik\cdot x}\} = A_\mu^+(x) + A_\mu^-(x) \tag{2.201}$$

We regard the components of the vector potential as independent quantities, thereby giving up the Lorentz condition. We then write the commutation relations

$$[a^\mu(\mathbf{k}), \mathrm{a}^{\dagger\nu}(\mathbf{k}')] = -g^{\mu\nu}\delta(\mathbf{k} - \mathbf{k}'), \qquad [a^{\dagger\mu}(\mathbf{k}), a^{\dagger\nu}(\mathbf{k}')] = [a^\mu(\mathbf{k}), a^\nu(\mathbf{k}')] = 0. \tag{2.202}$$

The a's and $a^\dagger$'s can be regarded as the annihilation and creation operators for four kinds of photons-two transverse, one longitudinal and one time-like. However, we face a difficulty in such a quantization. The time component a^0 satisfies the relation

$$[a^0(\mathbf{k}), a^{\dagger 0}(\mathbf{k}')] = -\delta(\mathbf{k} - \mathbf{k}'), \tag{2.203}$$

which shows that the creation and annihilation operators for the timelike photon behave as if they have interchanged places. This commutation relation is incompatible with the assumption that the metric in the Hilbert space of states is positive definite. The difficulty can be overcome by means of a formal approach due to Gupta-Bleuler based on the fact that the difficulty is due entirely to the timelike photons that do not actually exist, but the appearance of which in the intermediate steps of the argument is connected with the transition from observable quantities (**E**, **B**) to the unobservable four-potential A_μ.

Gupta-Bleuler formalism

In the Gupta-Bleuler formalism, the problems are overcome by modifying the vector space through a modification in the definition of the scalar product. Instead of

$$\langle \Phi, \Psi \rangle,$$

the scalar product is defined as

$$\langle \eta\Phi, \Psi \rangle,$$

where η is called a metric operator. The requirement of the norm to be real gives

$$\langle \eta\Phi \,|\, \Phi \rangle = \langle \eta\Phi \,|\, \Phi \rangle^* = \langle \Phi \,|\, \eta^\dagger \Phi \rangle^* = \langle \eta^\dagger \Phi \,|\, \Phi \rangle.$$

So

$$\eta = \eta^\dagger.$$

For convenience we take

$$\eta\eta^\dagger = \eta^2 = 1.$$

The expectation value of an operator is now

$$\langle \,|\, F \,|\, \rangle = \langle \eta\Phi \,|\, F \,|\, \Phi \rangle = \langle \Phi \,|\, \eta F \,|\, \Phi \rangle.$$

It is clear that $\langle F \rangle$ need no longer be real for Hermitian F. It will be real if $[\eta, F] = 0$. So η should be chosen so as to satisfy

$$[\eta, a^\lambda(\mathbf{k})] = 0, \qquad \lambda = 1, 2, 3.$$

The difficulty with timelike photons will be resolved if

$$[\eta, a^0(\mathbf{k})] = 0,$$

which can be satisfied with the choice

$$\eta = (-1)^{n_0}.$$

An important observation in the Gupta-Bleuler development was the recognition that the subsidiary condition $\partial_\mu A^\mu(x) \,|\, \Psi \rangle = 0$ for all x, on all the admissible states is too stringent. In fact, if such a condition is imposed as such there would be no states that would satisfy it. Gupta and Bleuler thus proposed that the Lorentz condition be modified so that it holds for the destruction part of the operator, namely, that

$$\partial^\mu A_\mu^+(x) \,|\, \Psi \rangle = 0 \quad \text{for all } x. \tag{2.204}$$

The weaker condition guarantees that the Lorentz condition is satisfied for the expectation values and is quite sufficient to establish correspondence with the classical field.

One consequence of this condition is that A_μ does not satisfy the wave equation as an operator equation, but its expectation value in any physically meaningful state does satisfy the wave equation. In order to understand the meaning of the new subsidiary condition, we go to the momentum space where the condition becomes

$$[a_3(\mathbf{k}) - a_0(\mathbf{k})]|\Phi\rangle = 0 \qquad \text{for all } \mathbf{k} \tag{2.205}$$

This is a constraint on the linear combination of longitudinal and scalar photons, for each value of k, which may be present in a state. It places no restriction on the transverse photons. If we calculate the expectation value of the energy of an allowed state, we find that due to this constraint, only transverse photons contribute to the energy. Same is true for all other observables.

Thus the result of the subsidiary condition is that in free space observable quantities involve only transverse photons. So the longitudinal and scalar photons are not observed as free particles. Only transverse photons are observed through the observables corresponding to the two degrees of freedom of the electromagnetic field which one finds in the Coulomb gauge.[1]

2.6 Electromagnetic Interactions

2.6.1 Electron-photon Interaction

In classical electrodynamics, the interaction Hamiltonian density of interaction between a current density j_μ and the electromagnetic field in the covariant form is given by

$$\mathcal{H}_{\text{int}} = j_\mu A_\mu. \tag{2.206}$$

In quantum electrodynamics, j_μ and A_μ are operators.

To describe the interaction of spin one-half particles with photon, the interaction Lagrangian density is taken as

$$\mathcal{L}_{\text{int}} = e : \bar{\psi}(x)\gamma^\mu\psi(x) : A_\mu(x) \equiv -\mathcal{H}_{\text{int}}, \tag{2.207}$$

charge of the particle (electron) is taken as $-e$. The current j_μ of electrons and positrons is constructed from the Dirac operators ψ and its adjoint $\bar{\psi}$. Each of these three operators ψ, $\bar{\psi}$ and A_μ have destruction and creation operators. The various terms in the interaction

$$(\bar{\psi}^{(+)} + \bar{\psi}^{(-)})\gamma^\mu(\psi^{(+)} + \psi^{(-)})(A_\mu^{(+)} + A_\mu^{(-)}) \tag{2.208}$$

can be represented diagrammatically as vertices. These vertices represent the fundamental processes which form the basic building blocks of the Feynman diagrams representing physical processes of the electromagnetic interaction.

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[1] We have not given the complete Gupta-Bleuler formalism here as for most purposes the complete formalism is not required. For more details see reference: S.N. Gupta (1977): *Quantum Electrodynamics*, Gordon and Breach, New York.

Consider $\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(-)}$, which destroys an electron, creates an electron and creates a photon, thereby representing an emission of a photon by an electron. In the momentum space the corresponding term $a^{\dagger}(\mathbf{k})c^{\dagger}(\mathbf{p}')c(\mathbf{p})$ when acting on a state that contains an electron of momentum **p**, replaces that state with one containing an electron of momentum **p**′ and an additional photon of momentum k. This process is depicted in Fig. 2.2. The straight line represents electron and the wavy line photon. We think of the diagram as a picture of the process in time with time running from bottom to the top of the page.

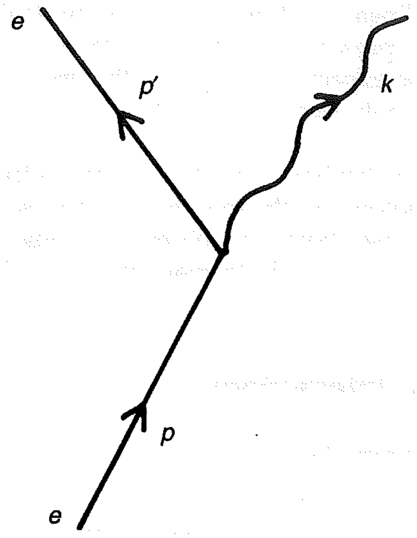


Fig. 2.2 Emission of a photon by an electron.

In the term $\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(+)}$ in the coordinate space or $a(\mathbf{k})c^{\dagger}(\mathbf{p}')c(\mathbf{p})$ in the momentum space, photon is annihilated and so is depicted by the diagram (Fig. 2.3) which shows the absorption of a photon by electron. Now consider the term $\bar{\psi}^{(+)}\gamma^{\mu}\psi^{(-)}A_{\mu}^{(-)}$ or its momentum

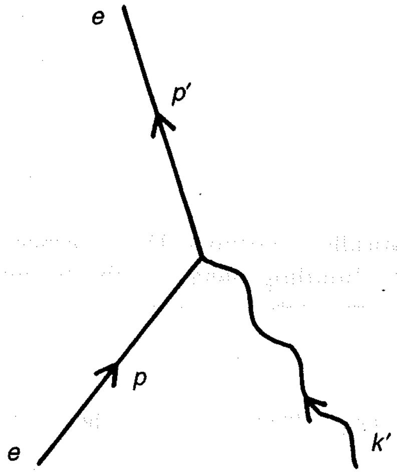


Fig. 2.3 Absorption of a photon by an electron.

transform $a^\dagger(\mathbf{k})d^\dagger(\mathbf{p}')d(\mathbf{p})$ representing the emission of a photon by positron. This term is represented by Fig. 2.4. Similarly, the term $\overline{\psi}^{(+)}\gamma_\mu\psi^{(-)}A_\mu^{(+)}$ represents a photon absorption by positron. The positron is represented by a line in the opposite direction. We now draw diagrams for pair creation and pair annihilation (Fig. 2.5) corresponding to the terms

$$\overline{\psi}^{(-)}\gamma_\mu\psi^{(-)}A_\mu^{(+)} \quad \text{and} \quad \overline{\psi}^{(+)}\gamma_\mu\psi^{(+)}A_\mu^{(-)}.$$

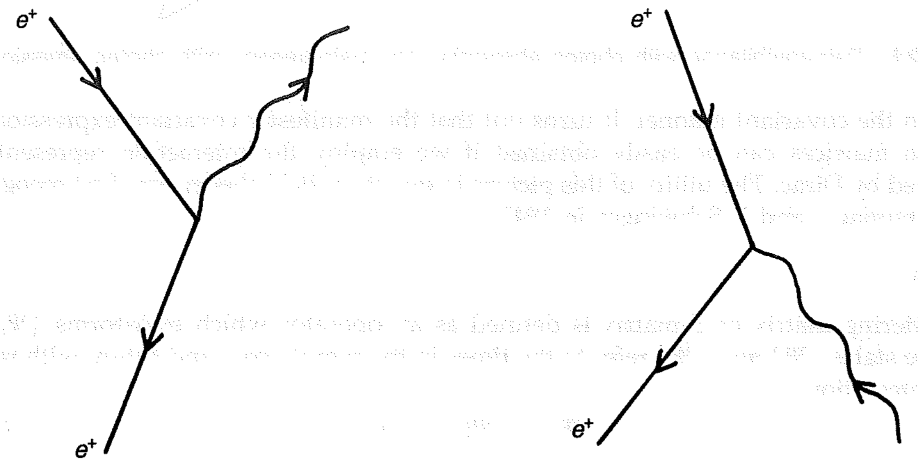


Fig. 2.4 Emission and absorption of photon by positron.

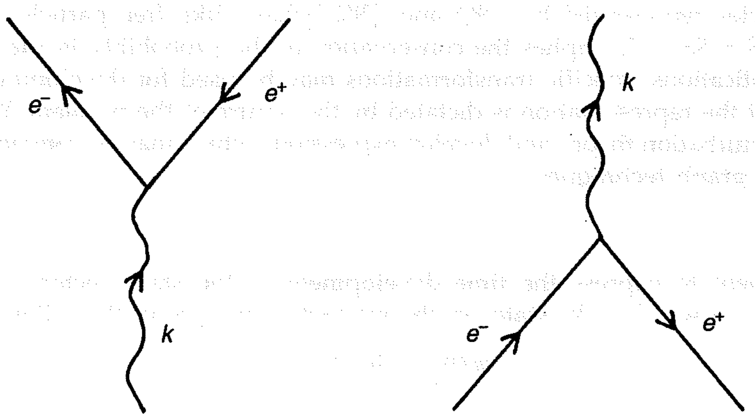


Fig. 2.5 Pair-creation and pair-annihilation.

Finally, we have the diagrams (Fig. 2.6) corresponding to terms $\overline{\psi}^{(+)}\gamma_\mu\psi^{(+)}A_\mu^{(+)}$ and $\overline{\psi}^{(-)}\gamma_\mu\psi^{(-)}A_\mu^{(-)}$ which in the momentum space are $a(\mathbf{k})c(\mathbf{p})d(\mathbf{p}')$ and $a^\dagger(\mathbf{k})d^\dagger(\mathbf{p})c^\dagger(\mathbf{p}')$.

2.7 Covariant Perturbation Theory

The usual time dependent perturbation theory is not very convenient for treating relativistic particles because the space time coordinates and the energy momentum variables do not

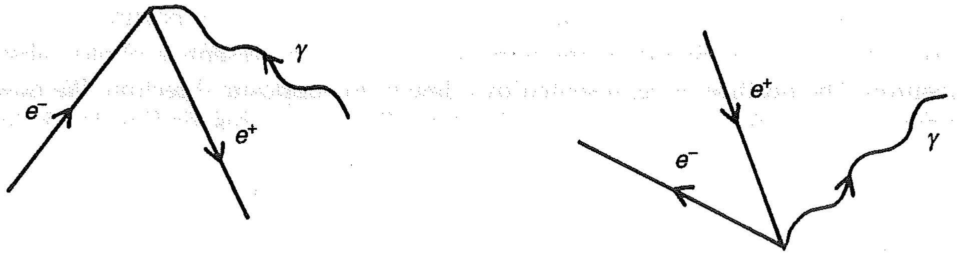


Fig. 2.6 Pair-annihilation with photon absorption and pair-creation with photon emission.

appear in the covariant manner. It turns out that the manifestly covariant expressions for transition matrices can be easily obtained if we employ the interaction representation introduced by Dirac. The utility of this picture in quantum field theory was first recognized by S. Tomonaga, and J. Schwinger in 1947.

S-matrix

The scattering matrix or S-matrix is defined as an operator which transforms $|\Psi_i\rangle$ into $|\Psi_f\rangle$. The states $|\Psi_i\rangle$ and $|\Psi_f\rangle$ refer to the times in the remote past and future with respect to the interaction.

$$|\Psi_i\rangle = |\Psi(t = -\infty)\rangle, \tag{2.209}$$

$$|\Psi_f\rangle = |\Psi(t = +\infty)\rangle. \tag{2.210}$$

So the particles represented by $|\Psi_i\rangle$ and $|\Psi_f\rangle$ behave like free particles. The S-matrix unitarity ($S^\dagger S = SS^\dagger = I$) implies the conservation of the probability in the transition. For practical applications, specific transformations may be used for developing the S-matrix. The choice of the representation is dictated by the nature of the problem. We will use the covariant perturbation theory and develop expressions which may be conveniently handled by Feynman graph techniques.

U-matrix

It is convenient to express the time development of the state vectors in terms of the operator $U(t, t_0)$ which links states in the interaction picture in the following manner:

$$|\psi_I(t)\rangle = U(t, t_0)|\psi_I(t_0)\rangle. \tag{2.211}$$

Then

$$S = \lim_{(t_0\to-\infty,\, t\to\infty)} U(t, t_0) = U(+\infty, -\infty). \tag{2.212}$$

Clearly, $U(t_0, t_0) = 1$, where 1 is the identity operator. From the equation of motion for $|\psi_I(t)\rangle$,

$$i\frac{\partial}{\partial t}|\Psi(t, t_0)\rangle = H_I(t)|\Psi(t, t_0)\rangle, \tag{2.213}$$

we deduce that

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t, t_0). \tag{2.214}$$

We can easily see that

$$U^{\dagger}(t, t')U(t, t') = U(t, t')U^{\dagger}(t, t') = \mathbf{1}, \tag{2.215}$$

and

$$U(t, t_1)U(t_1, t_0) = U(t, t_0). \tag{2.216}$$

2.7.1 Covariant Perturbation Theory

Take

$$U = \sum_{n=0}^{\infty} U_n = U_0 + U_1 + U_2 + U_3 + \ldots + U_{\infty}, \tag{2.217}$$

where the different terms are of the different orders of magnitude in H_I. The expansion is justified when H_I is small. Substituting in (2.214) and equating terms of equal order, we find

$$\left.\begin{aligned} i\frac{\partial}{\partial t} U_1 &= H_I, \quad \text{since } U_0 = 1 \\ i\frac{\partial}{\partial t} U_2 &= H_I U_1 \\ &\vdots \\ i\frac{\partial}{\partial t} U_{n+1} &= H_I U_n. \end{aligned}\right\} \tag{2.218}$$

Integrating (2.218), we obtain

$$\left.\begin{aligned} U_1(t, t_0) &= (-i/\hbar)\int_{t_0}^{t} H_I(t_1)\, dt_1, \\ U_2(t, t_0) &= (-i/\hbar)\int_{t_0}^{t} H_I(t_2)U_1(t_2)\, dt_2, \\ &= (-i/\hbar)^2 \int_{t_0}^{t} H_I(t_2)\, dt_2 \int_{t_0}^{t_2} H_I(t_1)\, dt_1. \end{aligned}\right\} \tag{2.219}$$

In general,

$$U_n = \left(\frac{1}{i}\right)^n \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \ldots \int_{t_0}^{t_2} dt_1 [H_I(t_n) H_I(t_{n-1}) \ldots H_I(t_1)].$$

Let us consider U_2. The first integral takes us from t_0 to t_2 along the x-axis and the second integral w.r.t. t_2 from t_0 to t along the y-axis (Fig. 2.7).

The region of integration may be represented by the vertically shaded triangle ABC below the line $t_1 = t_2$. Now if t_1 and t_2 are interchanged, we get

$$U_2' = (-i/\hbar)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} H_I(t_1) H_I(t_2)\, dt_2. \tag{2.220}$$

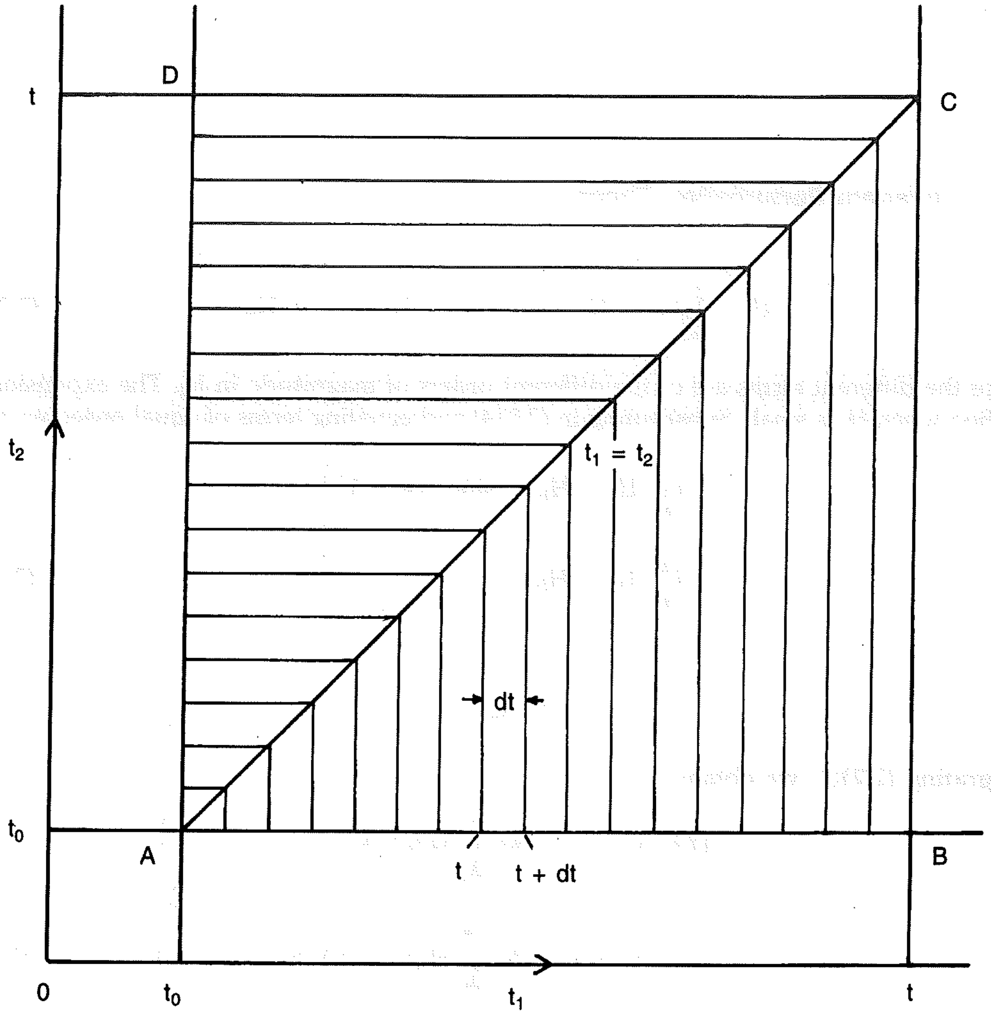


Fig. 2.7 Integration region.

The region of integration becomes horizontally shaded area ADC above the line. The integrand differs from the earlier one in the exchange of operators $H_I(t_1)$ and $H_I(t_2)$. If $H_I(t_1)$ and $H_I(t_2)$ commute the two integrands will be the same and $U_2 = U_2'$. U_2 can then be written as half of the integral over the whole square ABCD. Similar result can be achieved with non-commuting operators if we introduce the chronological product, introduced by Dyson who showed that the integration is essentially written as integration over the time interval from t_0 to t. The chronological operator P arranges any product of the time dependent operators in the chronological sequences. Thus we can write

$$U_2 = (-\,i/\hbar)^2\, \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\; P[H_I(t_1)H_I(t_2)], \tag{2.221}$$

where

$$P[H_I(t_1)H_I(t_2)] = \begin{cases} H_I(t_2)H_I(t_1), & t_1 < t_2 \\ H_I(t_1)H_I(t_2), & t_1 > t_2 \end{cases} \tag{2.222}$$

Similarly, we obtain, in general,

$$U_n = \left(\frac{1}{i}\right)^n \frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n \; P[H_I(t_1)H_I(t_2) \;\ldots\; H_I(t_n)]. \tag{2.223}$$

Therefore,

$$S = \sum_n \left(\frac{1}{i}\right)^n \frac{1}{n!} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \ldots \int_{-\infty}^{+\infty} dt_n P[H_I(t_1)H_I(t_2) \;\ldots\; H_I(t_n)]. \tag{2.224}$$

Now since

$$H_I(t) = \int d^3x \; \mathcal{H}_I(x), \tag{2.225}$$

we may write

$$S = \sum_n \left(\frac{1}{i}\right)^n \frac{1}{n!} \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n P[\mathcal{H}(x_n)\mathcal{H}(x_{n-1}) \;\ldots\; \mathcal{H}(x_1)]. \tag{2.226}$$

Given two state vectors $|\Psi_i\rangle$ and $|\Psi_f\rangle$, the amplitude for transition $\Psi_i \to \Psi_f$ is given by

$$(\Psi_f, S\Psi_i) \equiv \langle\Psi_f|S|\Psi_i\rangle = S_{fi}. \tag{2.227}$$

2.7.2 Applications of the S-matrix

We shall now apply the S-matrix formalism developed above to some physical problems. Consider first the simplest problem, the scattering of a relativistic electron by an external (unquantized) potential. This will illustrate some of the techniques and formalism of quantum field theory. In particular, we shall see how the matrix elements are calculated. The example will facilitate the subsequent formulation of the general rules of the invariant perturbation theory.

2.7.3 Scattering of Electrons by an External Field

In the presence of an external field represented by $A_\mu^{\text{ext}}(x)$, the interaction Hamiltonian density is written as

$$\mathcal{H}_{\text{int}}^{\text{ext}} = -\, e j^\mu(x) A_\mu(x). \tag{2.228}$$

The external field may be calculated in terms of the source density, i.e.

$$A_\mu^{\text{ext}} = \delta_{\mu 0}\phi(x), \tag{2.229}$$

with

$$\phi(x) = \frac{Ze}{4\pi|\mathbf{x}|}. \tag{2.230}$$

Ze is the strength of the point source at the origin. For a fixed source, momentum is not conserved; rather, the source can absorb or transfer momentum without changing its state. To compute the transition amplitude up to the first order from the initial state i to the final state f, we consider

$$S_{fi} = \langle f \,|- ie \int d^4x \bar{\psi}(x)\gamma_\mu \psi(x) A^\mu(x) \,|\, i\rangle. \tag{2.231}$$

This expression is the matrix element in the coordinate representation. In our case,

$$|\,i\rangle = \text{state of an electron with momentum } \mathbf{p} = c_r^\dagger(\mathbf{p}) \;|\,0\rangle, \tag{2.232}$$

$$|f\rangle = \text{state of an electron with momentum } \mathbf{p}' = c_{r'}^\dagger(\mathbf{p}') \,|\, 0\rangle. \tag{2.233}$$

And

$$\begin{aligned} \psi(x) &= \sum_{\mathbf{p}_1 r_1} \sqrt{\frac{m}{VE_1}}\, c_{r_1}(\mathbf{p}_1) u_{r_1}(\mathbf{p}_1) e^{-ip_1\cdot x} + \text{positron part} \\ &= \psi^{(+)} + \text{positron part}, \end{aligned} \tag{2.234}$$

$$\begin{aligned} \bar{\psi}(x) &= \sum_{\mathbf{p}_2 r_2} \sqrt{\frac{m}{VE_2}}\, c_{r_2}^\dagger(\mathbf{p}_2) \bar{u}_{r_2}(\mathbf{p}_2) e^{+ip_2\cdot x} + \text{positron part} \\ &= \bar{\psi}^{(-)} + \text{positron part}. \end{aligned} \tag{2.235}$$

The summation can also be replaced by integration for the continuum. We need to find

$$\langle 0 \,|\, c_{r'}(\mathbf{p}') c_{r_2}^\dagger(\mathbf{p}_2) c_{r_1}(\mathbf{p}_1) c_r^\dagger(\mathbf{p}) \,|\, 0\rangle, \tag{2.236}$$

as other combinations of operators will go to zero. Since the destruction operators acting from left on $|\,0\rangle$ and the creation operators acting from the right on $\langle 0\,|$ give us zero, we may like to rearrange the operators so that we move the destruction operators to the right and the creation operators to the left. In doing that we use the anticommutation rules for the annihilation and creation operators (2.162). We then have

$$c_{r_1}(\mathbf{p}_1) c_r^\dagger(\mathbf{p}) = \delta_{r_1 r} \delta^3(\mathbf{p}_1 - \mathbf{p}) - c_r^\dagger(\mathbf{p}) c_{r_1}(\mathbf{p}_1), \tag{2.237}$$

$$c_{r'}(\mathbf{p}') c_{r_2}^\dagger(\mathbf{p}_2) = \delta_{r_2 r'} \delta^3(\mathbf{p}_2 - \mathbf{p}') - c_{r_2}^\dagger(\mathbf{p}_2) c_{r'}(\mathbf{p}'). \tag{2.238}$$

The second term of these gives no contribution, and so we obtain

$$\langle 0 \,|\, c_{r'}(\mathbf{p}') c_{r_2}^\dagger(\mathbf{p}_2) c_{r_1}(\mathbf{p}_1) c^\dagger(\mathbf{p}) \,|\, 0\rangle = \delta_{r_1 r} \delta^3(\mathbf{p}_1 - \mathbf{p}) \delta_{r_2 r'} \delta^3(\mathbf{p}' - \mathbf{p}_2) \tag{2.239}$$

It is clear from this analysis that the positron parts of ψ and $\bar{\psi}$ will not contribute. The choice of the initial and final states picks up a specific term from the interaction. Replacing the summations over $\mathbf{p}_1$ and $\mathbf{p}_2$ by the integrations and performing the integrations we immediately obtain

$$S_{fi} = i\,\frac{e}{(2\pi)^3}\sqrt{\frac{m^2}{EE'}}\int d^4x\,\bar{u}_{r'}(\mathbf{p}')\gamma^0 u_r(\mathbf{p})\phi(x)e^{-i(p-p')\cdot x}. \tag{2.240}$$

The time integration is contained in

$$\int e^{-i(E-E')t}\,dt = 2\pi\delta\,(E-E'), \tag{2.241}$$

whereas the three-dimensional space integration gives

$$\int d^3x\,\frac{Ze^2}{4\pi|\mathbf{x}|}\,e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = \frac{Ze^2}{4\pi}\,\frac{4\pi}{|\mathbf{q}|^2}, \tag{2.242}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. So the S-matrix element becomes

$$S_{fi} = i\left(\frac{1}{2\pi}\right)^3 Ze^2\sqrt{\frac{m^2}{EE'}}\,\frac{\bar{u}_{r'}(\mathbf{p}')\gamma^0 u_r(\mathbf{p})}{|\mathbf{q}|^2}\,2\pi\delta(E-E'). \tag{2.243}$$

This is the expression for the matrix element in the momentum representation. We can represent it through a Fig. 2.8. From this, the differential cross-section can be found to be

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2}{|\mathbf{q}|^4}\,|\bar{u}_{r'}(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2, \tag{2.244}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. In the non-relativistic limit this gives the famous Rutherford formula.

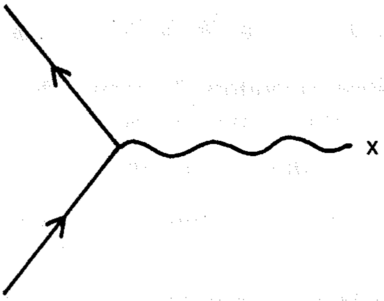


Fig. 2.8 Feynman diagram representing electron scattering by a potential.

2.7.4 Wick's Theorem

The treatment of higher order terms is more complicated. There exists a very powerful theorem, due to Wick, which enables us to express a chronological product of time dependent operators as a sum of products of normal ordered operators and allows us to treat S-matrix terms of various orders. There is also a simple graphical way, called Feynman diagram, of representing various terms in the S-matrix. We shall not discuss here all the details of Wick's theorem leading to Feynman diagrams but will present only some definitions of the concepts which are used.

2.7.5 Normal Product

The field variables ψ, $\overline{\psi}$, A_μ etc. appearing in $H_I(x)$ obey free field equations. They can be split into the positive and negative frequency parts. That is, they can be broken up into operators which destroy or create particles. The destruction operators remove particles present in the initial state $|\,\psi(-\infty)\rangle$ and creation operators create particles in the final state $|\,\psi(+\infty)\rangle$ which are not present in the initial state as was done in the previous section. Clearly, it is necessary to bring all the destruction operators to the right of all the creation operators, so that the destruction operators can act on the initial state. This can be achieved by putting different terms in the S-matrix series into normal products.

An operator O which is a product of free particle destruction and creation operators is called a *normal product* if all the creation operators stand to the left of all destruction operators. Let us introduce an operator N which, acting on a product of free particle operators rewrites them in the normal ordered form. The operator N includes the change of sign which may arise due to the rearrangement of the operators. For example,

$$\left.\begin{aligned} N(\phi^{(-)}(x)\phi^{(+)}(y)) &= N(\phi^{(+)}(y)\phi^{(-)}(x)) = \phi^{(-)}(x)\phi^{(+)}(y), \\ N(\psi^{(+)}(x)\,\overline{\psi}^{(-)}(y)) &= -\,N(\overline{\psi}^{(-)}(y)\psi^{(+)}(x)) = -\,\overline{\psi}^{(-)}(y)\psi^{(+)}(x), \\ N(A_\mu^{(+)}(x)A_\lambda^{(-)}(y)) &= N(A_\lambda^{(+)}(y)A_\mu^{(-)}(x)) = A_\mu^{(-)}(y)A_\lambda^{(+)}(x), \end{aligned}\right\} \tag{2.245}$$

In general, therefore,

$$N(C_r C_s^\dagger) \equiv \,:C_r C_s^\dagger:\, := \pm C_s^\dagger C_r. \tag{2.246}$$

where the upper sign is to be chosen when C, $C^\dagger$ are boson operators (and hence commuting) and the lower sign when these are fermion operators (and anticommuting).

$$N(\psi_\alpha^{(+)}\overline{\psi}_\beta^{(-)}\overline{\psi}_\gamma^{(+)}\psi_\delta^{(-)}) = +\psi_\delta^{(-)}\overline{\psi}_\beta^{(-)}\psi_\alpha^{(+)}\overline{\psi}_\gamma^{(+)} = -\,\overline{\psi}_\beta^{(-)}\psi_\delta^{(-)}\psi_\alpha^{(+)}\overline{\psi}_\gamma^{(+)}.$$

Consider a product of two linear operators. To reduce the product to the normal form at most one displacement of operators is needed. So we can write

$$A(x_1)B(x_2) =: A(x_1)B(x_2) : + \, C,$$

where C is a number. We shall call this number a contraction to be denoted by means of a line below the fields, i.e.

$$A(x_1)B(x_2) =: A(x_1)B(x_2) : + \underline{A(x_1)B}(x_2).$$

Since the vacuum expectation value of the normal product is always zero, we can write the contraction as the vacuum expectation value of the ordinary product, i.e.

$$\underline{A(x_1)B}(x_2) = \langle 0\,|\,A(x_1)B(x_2)\,|\,0\rangle.$$

As an example, consider a real scalar field

$$\begin{aligned} \phi(x)\phi(y) &= (\phi^{(+)}(x) + \phi^{(-)}(x))(\phi^{(+)}(y) + \phi^{(-)}(y)) \\ &= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y). \end{aligned} \tag{2.247}$$

Except for the second term all are in the normal form. Arranging it, we get

$$\begin{aligned}\phi(x)\phi(y) &= \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + i\Delta^{+}(x-y) \\ &\quad + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) \\ &= N(\phi(x)\phi(y)) + i\Delta^{+}(x-y), \end{aligned} \tag{2.248}$$

so that

$$\underbracket{\phi(x)\phi(y)} = \langle 0 \,|\, \phi(x)\phi(y) \,|\, 0\rangle \equiv i\Delta^{+}(x-y). \tag{2.249}$$

2.7.6 Physical Interpretation of Contraction

$\underbracket{A(x_1)B(x_2)}$ will vanish unless one of the operators creates a particle which the other operator annihilates. Therefore, the Feynman diagram (Fig. 2.9) corresponding to $\underbracket{A(x_1)B(x_2)}$ will be an internal line beginning at a vertex and ending at another vertex. It is referred to as Feynman propagator since it represents the creation of a (virtual) particle or an antiparticle at one vertex, its propagation to the other vertex and its annihilation there. Since the contraction is just the vacuum expection value of the product of operators, the contribution to the matrix element in the coordinate representation of an internal line is

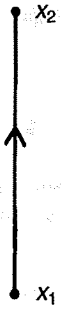


Fig. 2.9 Feynman propagator.

$$\underbracket{A(x_1)B(x_2)} = \langle 0 \,|\, A(x_1)B(x_2) \,|\, 0\rangle,$$

which for the real scalar field is $i\Delta^{+}(x-y)$. We can obtain corresponding expressions for electron, positron and charged scalar particle propagators. As we have seen, the propagator is obtained from the commutator or anticommutator of the corresponding destruction and creation operators.

Covariant commutation relations

$$\begin{aligned} i\Delta^{+}(x-y) &= [\phi^{(+)}(x), \phi^{(-)}(y)] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}}\,\frac{d^3k'}{\sqrt{2\omega_{k'}}}\, e^{-ik\cdot x} e^{ik'\cdot y} [a_k, a_{k'}^{\dagger}] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, e^{-ik\cdot(x-y)} \end{aligned} \tag{2.250}$$

so that

$$\Delta^{+}(x) = \frac{-i}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{-ik\cdot x}, \tag{2.251}$$

which in the covariant form is equal to

$$\Delta^{+}(x) = \frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty} d^4k\ \theta(k_0)\delta\,(k^2 - m^2)e^{-ik\cdot x}, \tag{2.252}$$

$\theta(k_0)$ is defined by

$$\theta(k_0) = \begin{cases} 1 & \text{if } k_0 > 0 \\ 0 & \text{if } k_0 < 0 \end{cases}$$

The equivalence of (2.251) and (2.252) is established by recalling that

$$\delta(k^2 - m^2) = \frac{1}{2|k|}\,[\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)], \tag{2.253}$$

where

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2},$$

and recognizing that the $\theta(k_0)$ factor implies only the $\delta(k_0 - \omega_k)$ contributes. $\delta(k^2 - m^2)$ asserts that k^2 is a time-like vector. The $\theta(k_0)$ then implies that k lies in the forward light cone. Similarly,

$$[\phi^{(+)}(x), \phi^{(-)}(y)] = i\Delta^{-}(x - y), \tag{2.254}$$

where

$$\Delta^{-}(x - y) = \frac{i}{2(2\pi)^3}\int \frac{d^3k}{\omega_k}\, e^{ik\cdot x}, \tag{2.255}$$

which in the covariant form is equal to

$$\Delta^{-}(x) = \frac{i}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k\theta(k_0)\delta(k^2 - m^2)\, e^{ik\cdot x}$$

$$= \frac{i}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k\theta(-k_0)\delta(k^2 - m^2)\, e^{-ik\cdot x}. \tag{2.256}$$

Therefore,

$$\Delta^{+}(x) = -\Delta^{-}(-x). \tag{2.257}$$

Now

$$\Delta(x) = \Delta^{+}(x) + \Delta^{-}(x), \tag{2.258}$$

and so

$$\Delta(x) = -\frac{i}{2(2\pi)^3}\int_{k_0>0} \frac{d^3k}{\omega_k}\,(e^{-ik\cdot x} - e^{ik\cdot x}),$$

$$\Delta(x) = -\frac{1}{(2\pi)^3}\int_{k_0>0}\frac{d^3k}{\omega_k}\,e^{ik\cdot x}\sin(k_0x_0),$$

$$= \frac{-i}{(2\pi)^3}\int_{-\infty}^{+\infty} d^4k\varepsilon(k_0)\delta(k^2-m^2)e^{-ik\cdot x}, \tag{2.259}$$

where

$$\varepsilon(k_0) = \theta(k_0) - \theta(-k_0)$$

$$= \frac{k_0}{|k_0|} = \begin{cases} +1 & \text{for } k_0 > 0 \\ -1 & \text{for } k_0 < 0 \end{cases} \tag{2.260}$$

$\Delta(x)$ is defined through

$$[\phi(x), \phi(y)] = \frac{1}{(2\pi)^3}\int\frac{d^3k}{\sqrt{2\omega_k}}\,\frac{d^3k'}{\sqrt{2\omega_{k'}}}\,\{e^{-ik\cdot x+ik'\cdot y}[a_k, a_{k'}^+] + e^{ik\cdot x-ik'\cdot y}[a_k^+, a_{k'}]\}$$

$$= i\Delta(x-y). \tag{2.261}$$

so that

$$\Delta(x) = \frac{-i}{2(2\pi)^3}\int\frac{d^3k}{\omega_k}(e^{-ik\cdot x} - e^{ik\cdot x}).$$

It is easy to see that

$$[\phi^{(+)}(x), \phi^{(+)}(y)] = 0, \qquad [\phi^{(-)}(x), \phi^{(-)}(y)] = 0 \tag{2.262}$$

and

$$\left.\frac{\partial\Delta(x)}{\partial x_0}\right|_{x_0=0} = -\frac{1}{(2\pi)^3}\int_{k_0>0}\frac{d^3k}{\omega_k}\,e^{ik\cdot x} = -\delta(x). \tag{2.263}$$

Dirac field

$$[\psi^{(+)}(x), \overline{\psi}^{(-)}(y)] = \frac{1}{(2\pi)^3}\int d^3p d^3p'\sqrt{\frac{m^2}{E_pE_{p'}}}\sum_{r,s}e^{-ip\cdot x+ip'\cdot y}\overline{u}_r(\mathbf{p})u_s(\mathbf{p})[c_r(\mathbf{p}), c_s^+(\mathbf{p}')]$$

$$= \frac{1}{(2\pi)^3}\int d^3p\left(\frac{m}{E_p}\right)\sum_r u_r(\mathbf{p})\overline{u}_r(\mathbf{p})e^{-ip\cdot(x-y)}$$

$$= \frac{1}{(2\pi)^3}\int d^3p\left(\frac{m}{E_p}\right)\frac{\gamma\cdot p+m}{2m}\,e^{-ip\cdot(x-y)} \tag{2.264}$$

$$= \frac{1}{(2\pi)^3}\int d^3p\left(\frac{m}{E_p}\right)(i\gamma\cdot\partial + m)\,e^{-ip\cdot x}$$

$$= (i\gamma \cdot \partial + m)\, \frac{1}{(2\pi)^3} \int d^3p \left(\frac{m}{E_p}\right) e^{-ip\cdot x} \tag{2.265}$$

$$= i(i\gamma \cdot \partial + m)\Delta^{(+)}(x). \tag{2.266}$$

2.8 Feynman Graphs and Their Applications

2.8.1 The Configuration-space Rules

Feynman generalized the rules and these are:

To calculate the nth order contribution, we draw all distinct graphs connecting the points; each point has an incoming fermion line, an outgoing fermion line, and a photon line attached to it. Only connected graphs are to be drawn. The contribution from each graph is calculated as follows:

- For each contracted pair of fields $\psi(x)\,\bar{\psi}(y)$, we draw a directed solid line from x to y, the direction of charge flow being indicated by an arrow on the line.
- For each $\bar{\psi}(x)$ factor, which is not contracted, we draw a directed line from x with the other end being free.
- For each $\psi(x)$, which is not contracted, we draw a solid line from bottom edge of the diagram to the point x which is a vertex.
- For each $A_\mu(x)$ we draw a wavy line from x, the other end being free.

2.8.2 Momentum-space Rules

It is much simpler to work in momentum space. Actually, the integration of the S-matrix elements with respect to the coordinates leads immediately to the expression in the momentum space and hence to the rules in the momentum space.

In the momentum space, we label each of the lines with a momentum. At each vertex there will appear a Dirac delta function representing four momentum conservation. There is also associated with the momentum conserving delta function a factor of $(2\pi)^4$. There is a factor i for each order of the perturbation theory, an e for each vertex and an i for each internal line.

For each external line, we write $u(p)$ for an incoming electron line, $\bar{v}(p)$ for an outgoing positron line (here the incoming line will have an arrow pointing away from the graph, and momentum flowing opposite to the arrow direction) $\bar{u}(p')$ for an outgoing electron line, $v(p')$ for an outgoing positron line, $\varepsilon_\mu^\lambda(k)$ for an external photon of polarization λ entering a vertex with γ^μ in it.

At each vertex a factor $-ie\gamma^\mu$ is written, together with a delta function representing four-momentum conservation at the vertex.

For each internal photon line labelled with momentum k, we write

$$-\frac{g_{\mu\nu}}{k^2 + i\varepsilon},$$

Here μ and ν are the indices on the Dirac matrices at the vertex connected by the internal photon line.

For each internal fermion line a factor

$$\frac{1}{\gamma\cdot p - M + i\varepsilon} = \frac{\gamma\cdot p + M}{p^2 - M^2 + i\varepsilon}$$

is written, the momentum being chosen to flow in the same direction as the charge, i.e. in the direction of the arrow.

All internal momenta are integrated over. This leads to an overall four momenta conserving delta function for the external momenta.

Each term is to be multiplied by $(i)^{n+f_i+b_i}$, where f_i is the number of internal fermion lines and b_i is the number of internal photon lines. Furthermore, each term is multiplied by $(2\pi)^{4(n-f_i-b_i)}(2\pi)^{-3/2(f_e-b_e)}$, where f_e is the number of external fermion lines and b_e is the number of external photon lines.

The contribution from the graph must be multiplied by ±1, the signature of the permutation of the fermions in the final state.

There is an additional factor $(-1)^L$, where L is the number of closed loops in the graph. This comes about because closed loops correspond to terms involving $\langle 0\,|\,j_\mu j_\nu \dots j_\lambda\,|\,0\rangle$. The rearrangement of $\bar\psi$ and ψ operators to yield the $\psi\bar\psi$ ordering (which leads to the appearance of $-\frac{1}{2}S_F(x_i - x_j)$) always involves an odd number of permutations.

The rules do not yet include the effect of mass renormalization—the inclusion of $\delta m\,\bar\psi\,\psi$ in the interaction Lagrangian or the effect of multiplying the matrix element by the still undetermined renormalization constant.

The rules also need modification when some of the integrals turn out to be infinite.

2.8.3 Scattering of a Photon by an Electron (Compton Scattering)

Let us consider a process in which a photon is scattered by an electron.

$$\gamma + e \rightarrow \gamma + e.$$

Let k and p(k' and p') be the four momenta of the initial (final) photon and electron respectively, r and r' spin projections of the initial and final electron, and λ and λ' the polarization vectors of the initial and final photon. The initial and final state vectors are

$$|p, r; k, \lambda\rangle = c_r^\dagger(\mathbf{p})\,a_\lambda^\dagger(\mathbf{k})\,|\,0\rangle,$$

$$|p', r'; k', \lambda'\rangle = c_{r'}^\dagger(\mathbf{p}')\,a_{\lambda'}^\dagger(\mathbf{k}')\,|\,0\rangle.$$

Let us look at the matrix element

$$\langle p', r'; k', \lambda'\,|\,S\,|\,p, r; k, \lambda\rangle. \tag{2.267}$$

It is easy to see that $S^{(1)}$ does not contribute to this process. The lowest order term that will contribute to this process is $S^{(2)}$. The relevant matrix elements of $S^{(2)}$ correspond to Fig. 2.10. Starting from the final state and using the Fenyman rules in momentum space, we obtain for the matrix element of Compton effect in the second order perturbation theory, the relation

$$S_{fi} = (-i)^2 \varepsilon^*_\mu(\mathbf{k}')\bar{u}(p')e\gamma^\mu \left\{\frac{\gamma\cdot(p+k)}{(p+k)^2 - m^2}\right\} e\gamma^\nu u(p)\, \varepsilon_\nu(\mathbf{k})\,\delta^4(p' + k' - p - k)$$

$$+ (-i)^2 \varepsilon_\nu(\mathbf{k}')\bar{u}(p')e\gamma^\nu \left\{\frac{\gamma\cdot(p-k)}{(p-k)^2 - m^2}\right\} e\gamma^\mu u(p)\, \varepsilon^*_\mu(\mathbf{k})\,\delta^4(p' + k' - p + k).$$

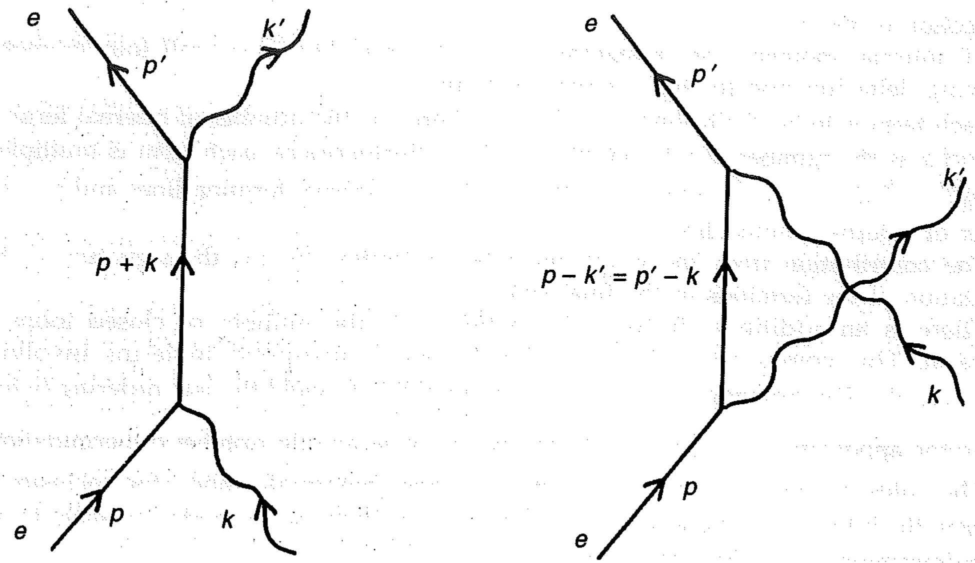


Fig. 2.10 Compton scattering.

2.8.4 Electron-Electron Scattering

We now consider the process of electron scattering by a free electron

$$e + e \rightarrow e + e.$$

Let p_1, p_2, and p'_1, p'_2 be the four momenta of the initial and final electrons respectively, r_1, r_2 and r'_1, r'_2 their spin projections. The initial and final state vectors are

$$|p_1, r_1; p_2, r_2\rangle = c^\dagger_{r_1}(p_1)c^\dagger_{r_2}(p_2)\,|\,0\rangle,$$

$$|p'_1, r'_1; p'_2, r'_2\rangle = c^\dagger_{r'}(p'_1)c^\dagger_{r'}(p'_2)\,|\,0\rangle.$$

We need to evaluate

$$\langle p'_1, r'_1; p'_2, r'_2|S|p_1, r_1; p_2, r_2\rangle. \tag{2.268}$$

Again, it is obvious that the operator $S^{(1)}$ does not contribute to the matrix element of the process. For the second order terms, in the one photon exchange approximation the Feynman diagrams are given in Fig. 2.11. The matrix element is

$$S_{fi} = e^2\bar{u}(p'_1)\gamma_\mu u(p_1)\,\frac{1}{(p'_1 - p_1)^2}\,\bar{u}(p'_2)\gamma^\mu u(p_2) - e^2\bar{u}(p'_2)\gamma_\mu u(p_1)\,\frac{1}{(p'_2 - p_1)^2}\,\bar{u}(p'_1)\gamma^\mu u(p_2),$$

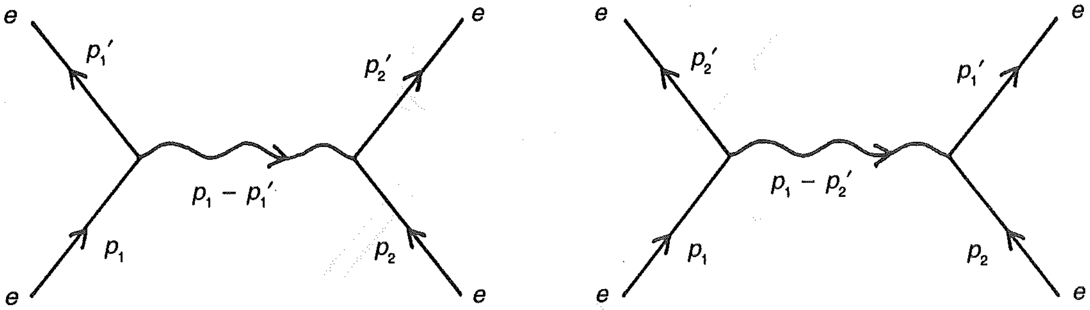


Fig. 2.11 Electron-electron scattering.

The minus sign between the two graphs is due to the crossed fermion lines in the second diagram (i.e. due to the exchange of electrons in the final state).

2.8.5 Electron-muon Scattering

One can treat the electron-muon scattering in a manner similar to the electron-electron scattering except that we will not have the exchange scattering as the projectile and target are different particles. In the lowest order (second order in the case under consideration) of perturbation theory—the one photon exchange approximation—the relevant Feynman diagram is given in Fig. 2.12. The matrix element is written according to the Feynman rules and is

$$S_{fi} = e^2 \bar{u}(p_1') \gamma_\mu u(p_1) \frac{1}{(p_1' - p_1)^2} \bar{u}(p_2') \gamma^\mu u(p_2). \tag{2.269}$$

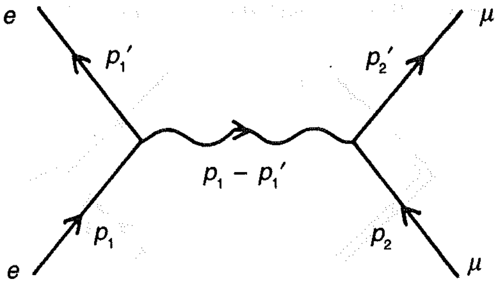


Fig. 2.12 Electron-muon scattering.

Electron-proton scattering

The electron-proton scattering (Fig. 2.13) is similar to the electron-muon scattering except that the photon proton-vertex will be different from the corresponding vertex for muon because, whereas the muon is a point particle, the proton has a structure.

$$S_{fi} = e^2 \bar{u}(p') \gamma_\mu u(p) \frac{1}{(p' - p)^2} \langle P' | j^\mu | P \rangle. \tag{2.270}$$

We have so far considered only the diagrams corresponding to the lowest order

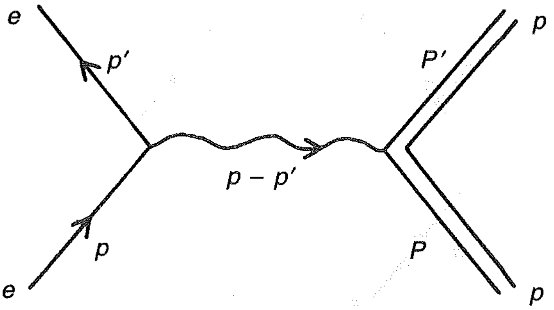


Fig. 2.13 Electron-proton scattering.

contribution (tree level) of the S-matrix to a process. For higher order of perturbation series, we come across divergences in the loop integrals. These divergences are eliminated through renormalization.

2.9 Higher Order Corrections

The measurement in 1947 of the Lamb shift was a memorable event which stimulated the development of quantum field theory, particularly, of the Quantum Electrodynamics (QED), the quantum field theory of electromagnetic interactions. The explanation of the Lamb shift was provided by H. Bethe, R.P. Feynman, J. Schwinger, S. Tomonaga ~~and others~~ in the framework of QED. The shift arises due to radiative corrections. A process which receives contribution in the lowest order from $S^{(2)}$ can also get contributions from higher orders in perturbation theory. For example, the electron-proton scattering has fourth order contributions coming from Fig. 2.14. These diagrams here are called loop diagrams and their contributions are named radiative corrections. Formal calculation of these higher order diagrams shows

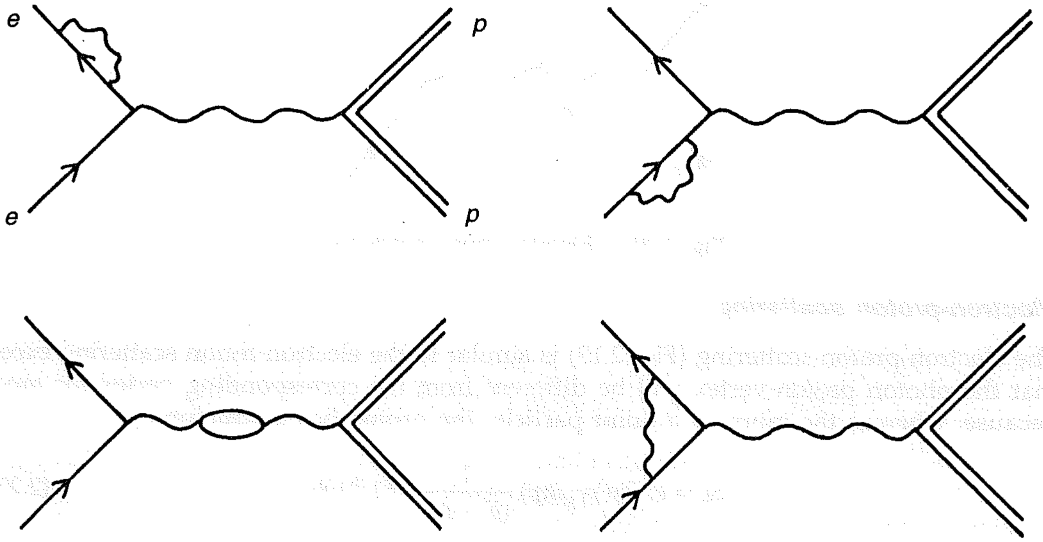


Fig. 2.14 Higher order corrections.

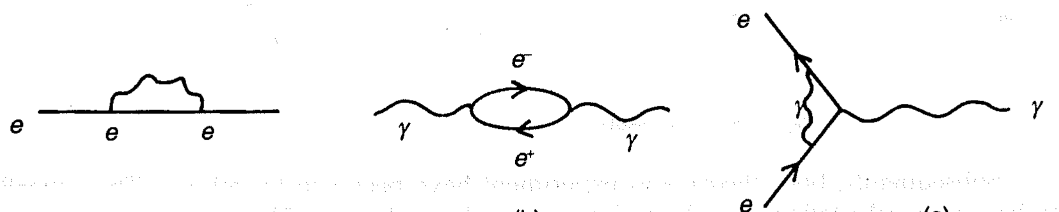
that they yield infinite contributions. A careful investigation shows that these diagrams are due to the three basic processes described by Fig. 2.15.

(a) Corrections to an external electron line or internal electron line (electron propagator)
(b) Correction to a photon external line or an internal line (photon propagator)
(c) Correction to a vertex.

The infinities due to the radiative corrections are removed with the help of a process called renormalization as a result of which finite corrections are obtained. By 'renormalization' we mean, together with the redefinition of the mass and coupling constant, the readjustment of the normalization of propagator by suitable multiplicative factors which may eliminate possible infinities in the propagator. At each order of perturbation theory, the process of removing divergences turns out to be a subtraction of divergent pieces in the proper (one particle irreducible) propagator. The central subject of a renormalization theory lies in the question whether the above subtraction process is consistently performed to all orders with a finite number of multiplicative factors (renormalization constants) and parameters which are redefined. Renormalization programme is successfully carried out only in a restricted class of interactions. The theory for which the above renormalization procedure works is said to be renormalizable.

To see how the renormalization is done, let us first consider Fig. 2.15(a), which is called the electron self-energy, because it has the effect of changing the energy of the electron which is equivalent to change in the rest mass δm. Calculation shows that it is infinite. The parameters appearing in the Lagrangian such as masses and coupling constants are not measurable quantities. We, must for example, add to a bare mass (m_0) an electromagnetic interaction contribution (δm) to obtain the physical (observable) mass m. The observable mass is finite but the bare mass can be infinite. So the effect of the inclusion of self energy is to replace in the Lagrangian the infinite bare mass $m_0(= m - \delta m)$ by the finite observable mass m. This procedure is called mass renormalization.

Charge renormalization is required when we consider the effect of Fig. 2.15(b). This effect is called vacuum polarization and it multiplies the unobservable bare coupling constant e_0 by an infinite factor to give a finite observable coupling constant e. In charge renormalization, we redefine the electron charge so as to absorb the infinite factor.



(a) (b) (c)

Fig. 2.15 Basic processes of higher order corrections.

Let us first consider the so-called radiative corrections in the lowest order of perturbation theory. The Feynman diagrams corresponding to the radiative corrections to a process contain in addition to the diagram describing the process in the lowest order, the vertices corresponding to the emission and reabsorption of virtual photons. For the second order radiative corrections, we will then expect corrections of the order of fine structure constant, α to be the lowest order result.

The program we have outlined can be carried out to all orders of perturbation theory, so that the radiative corrections can be calculated to a high degree of accuracy. An extraordinary agreement of these predictions with equally precise experimental data for the anomalous magnetic moment of the electron and for the Lamb shift constitute the most spectacular success of modern theoretical physics.

2.9.1 The Anomalous Magnetic Moment of the Electron

Let us now consider the simplest problem involving radiative corrections, namely, the calculation of the anomalous magnetic moment of the leptons (electron and muon). According to the Dirac theory, the electron possesses a magnetic moment

$$\mu = (-e/2M).$$

The magnetic moment is measured in the interaction of electron with an external electromagnetic field. The electron magnetic moment is modified due to the second order radiative corrections (Fig. 2.16). They were first derived by Schwinger in 1948. The result

$$\delta\mu = \frac{1}{2\pi}\left(\frac{e^2}{4\pi}\right)\mu = .0011614\ \mu$$

was in excellent agreement with the experimental value

$$\delta\mu = (.001165 \pm .000011)\mu$$

corresponding to a shift in the g-factor

$$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} = .00116.$$

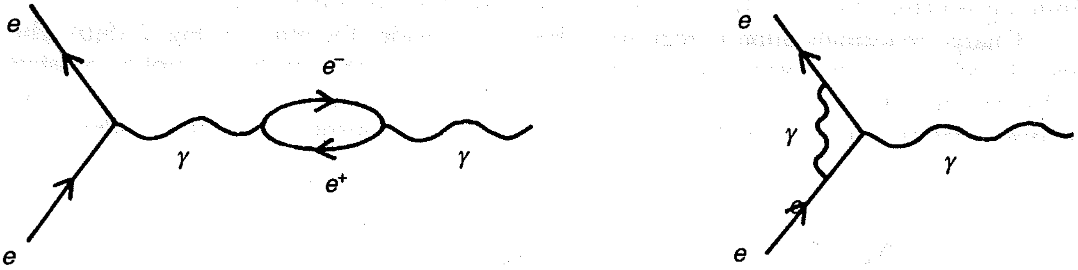


Fig. 2.16 Anomalous magnetic moment of electron.

Subsequently, both theory and experiment have been greatly refined. Theoretically, higher order calculations to a high degree have been derived. The result is

$$10^9 a_e = 1159652.4 \pm 0.4.$$

The most recent experiment by Van Dyck et al. (1978) gives

$$10^9 a_e = 1159652.41 \pm 0.2.$$

For muon, the theoretical value is

$$a_\mu = .00116,$$

and the experimental value

$$10^9 a_e = 116592 \pm 8.5.$$

2.9.2 Lamb Shift

Some loop diagrams contributing to the Lamb shift are given in Fig. 2.17. Bethe was the first to give in 1947 an approximate non-relativistic derivation of the Lamb shift due to the self energy of electron in a bound state. Because of the great importance of the Lamb shift as a test of QED, both calculations and measurements of Lamb shift have been greatly refined. The calculations take into account higher order corrections, small effects due to finite size and finite mass of the nucleus and use of the relativistic wavefunctions for the bound state of the hydrogen atom. The most recent calculations give 1057.916 ± .016 MHz, a value to be compared with the most recent experimental data, 1057.893 ± 0.02 MHz. The agreement is remarkable.

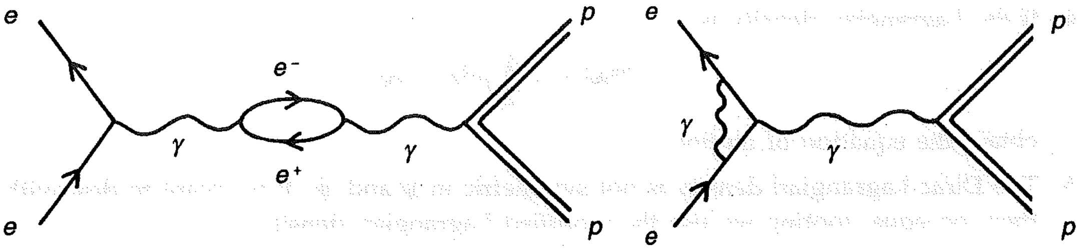


Fig. 2.17 Higher order corrections contributing to Lamb shift.

Summary

Path integral quantization

In this chapter, we have discussed only canonical quantization, which we use in the remaining part of the book. There is also an alternative to canonical quantization that is based on early ideas of Dirac, but was developed primarily by Feynman. Path integral quantization is equivalent to canonical quantization, but for many applications the path integral approach is more convenient. In particular, quantization of non-abelian gauge theories is accomplished most beautifully through path integral (PI). The path integral approach has the advantage of exhibiting a closer relationship to classical description and the manipulations involve only ordinary functions.

PROBLEMS

1. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$; show that γ^5 is Hermitian and $(\gamma^5)^2 = 1$.
2. Obtain the equation(s) of motion for the field whose Lagrangian density is

given by

$$\mathcal{L} = i\psi^*(\mathrm{x}, t)\ \dot{\psi}(\mathrm{x}, t) - \frac{\hbar^2}{2m}(\nabla\psi^*)\cdot(\nabla\psi) - V(\mathrm{x})\psi^*(\mathrm{x}, t)\psi(\mathrm{x}, t),$$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \lambda(\phi^*\phi)^2.$$

3. Obtain the equation of motion for the field ϕ whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \lambda\phi^4.$$

4. If the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \rho\phi$$

obtain the equation of motion.

5. The Dirac Lagrangian density is not symmetric in ψ and $\bar{\psi}$. If we want to deal with them on equal footing we use the modified Lagrangian density

$$\mathcal{L} = \frac{ic}{2}\left[\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - mc^2\bar{\psi}\psi.$$

To find from the Euler-Lagrange equation, the Dirac equation and its adjoint, use the former equation.

6. If $N_r^{(+)} = C_r^\dagger C_r$, where C_r, C_s^+ satisfy canonical anticommutation relations, show that

$$[N_r^{(+)}, C_s] = -\delta_{rs}C_r, \quad [N_r^{(+)}, C_s^\dagger] = \delta_{rs}C_s^\dagger.$$

What do you infer from these relations about the physical significance of the operators C_r and C_r^+?

7. Show that $N(\mathbf{k}) = C^\dagger(\mathbf{k})C(\mathbf{k})$ satisfies the relation $N^2(\mathbf{k}) = N(\mathbf{k})$ if $C(\mathbf{k})$ and $C^\dagger(\mathbf{k})$ obeys canonical anticommutation relations. Give the physical meaning of this equation.

8. Deduce the commutation rules for $a(\mathbf{k})$ and $a^+(\mathbf{k}')$ if

$$a(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2k_0}}(k_0\phi + i\pi),$$

$$a^\dagger(\mathbf{k}') = \frac{1}{(2\pi)^{3/2}}\int d^3x\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2k_0'}}(k_0'\phi - i\pi),$$

where $\phi(x)$ and $\pi(x)$ are the real scalar field and its canonical conjugate momentum field respectively.

9. For the complex field:

(i) Show that $a(-\mathbf{k}) = b^\dagger(\mathbf{k})$, and $b(-\mathbf{k}) = a^\dagger(\mathbf{k})$

(ii) Using (i) prove that

$$\int d^3x\dot{\phi}^*(x)\dot{\phi}(x) = 0$$

and

$$\int d^3x\phi^*(x)\phi(x) = \int d^3k\frac{1}{w_k}\{N^{(+)}(\mathrm{k}) + N^{(-)}(\mathrm{k}) + 1\}$$

and hence show that the total Hamiltonian of the complex field can be written as

$$H = \sum_{\mathrm{k}} [N^{(+)}(\mathrm{k}) + N^{(-)}(\mathrm{k})]\omega_k.$$

10. Write the Feynman amplitude for electron-electron scattering in the lowest order.
11. Write the amplitude for the scattering of electron by a proton.
12. Using the Lorentz covariance, show that the most general form of the matrix element of electromagnetic current between the two proton states is

$$\langle P'|j_\mu(0)|P\rangle = \bar{u}(P')[\gamma_\mu F_1(q^2) + i\sigma_{\mu\nu}q^\nu F_2(q^2)]u(P).$$

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*Almost all books on gauge field theories deal with path integral quantization.

3

SYMMETRIES

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through ages has tried to comprehend and create order, beauty and perfection.

H. Weyl

The concept of symmetry has been of ever increasing importance in fundamental theoretical physics. The symmetry principles are often the only guides into the unknown domain, and in the realm of the known they provide us with the convenient short cuts in the calculations of important results. In fact, since the beginning of physics, symmetry considerations have provided us an extremely useful and powerful tool to understand nature. It has been so because they are general statements made about systems and are not related to any specific dynamical model. Any dynamical model, of course, must conform to the symmetry principles observed. In fact, the existence of certain symmetries provides us the path to choose the right model saving a lot of time and energy by avoiding useless models. Finally, the amount of effort expended on evaluating the consequences of a model can be reduced if the more general results of the overlying symmetry can be applied. For example, we do not have as yet a complete dynamical theory of strong interactions, and 'symmetries and conservation laws', therefore, play a central role in nuclear and particle physics. They help us in classifying the observed phenomena and states and also in predicting new features. They can serve as guides towards a future dynamical theory. In recent years, a greatly expanded role of symmetry principles has come about through the concept of gauge invariance.

Many physical systems have exact or approximate symmetries. The electron, for instance, moves in a spherically symmetric potential in atoms when spin is ignored. The nuclear forces between proton and neutron, between proton and proton and between neutron and neutron are the same leading to isospin symmetry in nuclei. The electron in a crystal moves in a field with periodic structure and so with certain symmetry. The general relationship between symmetries and dynamics will help us in the understanding of the properties of such physical systems.

In order to see the power of invariance, let us spend some time in setting up the formal relations between symmetries, conservation laws and generators of transformations.

In classical physics, the connection between invariance principle and conservation laws can be established using Poisson brackets or with the help of Noether's Theorem. For example, we know from classical mechanics that invariance under symmetry leads to conservation law, e.g.

(a) invariance under spatial translation ⇒ momentum conservation,
(b) invariance under time translation ⇒ energy conservation,
(c) invariance under rotation ⇒ angular momentum conservation.

3.1 Conservation Laws in Quantum Mechanics

The translation of the connection between symmetries and conservation laws to quantum mechanics is straightforward and is well known. If H is the Hamiltonian of a system and F a Hermitian operator corresponding to a physical observable, then

$$\frac{dF}{dt} = \frac{-i}{\hbar}[F, H]. \tag{3.1}$$

The observable represented by the operator F is a constant of motion (conserved quantity) if F commutes with the Hamiltonian and does not depend explicitly on time. Observables of this type have a practical value. Since they commute with Hamiltonian, a suitable set of them together with H itself may be taken as the set of compatible observables (commuting operators), in terms of which the states of any system may be specified. If the Hamiltonian is known, the question is: what are the operators that correspond to conserved quantities? To answer this question the behaviour of the Hamiltonian under various transformations is studied. We study transformations that are continuous and transformations that are discrete.

Let U be a linear operator representing the symmetry transformation of a physical system. We may have seen earlier that U must be a unitary operator.[1] We briefly review in the following, the argument to show that U must be unitary.

Consider a complete set of orthonormal states of a physical system denoted by $|m\rangle$. The set may be taken to be discrete; so,

$$\langle m | n\rangle = \delta_{m,n}. \tag{3.2}$$

Linear transformation takes this set to a new set of states $|m'\rangle$. That is,

$$|m'\rangle = U|m\rangle, \tag{3.3}$$

$$\langle m'| = \langle m|U^{\dagger}. \tag{3.4}$$

To preserve orthonormality, we must have

$$\langle m'|n'\rangle = \langle m|U^{\dagger}U|n\rangle = \langle m|n\rangle \tag{3.5}$$

which requires

$$U^{\dagger}U = 1. \tag{3.6}$$

We are, therefore, restricted to the unitary transformations.

Let us now consider an operator A representing a physical observable. It may be the

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[1] We shall see later in this chapter that antiunitary transformation may also be used to describe some symmetry transformation.

Hamiltonian operator, momentum operator, charge operator, the S-matrix, the transition operator T, etc. We can write any matrix element of A as

$$\langle m|A|n\rangle = \langle m|U^{\dagger}UAU^{\dagger}U|n\rangle = \langle m'|A'|n'\rangle \tag{3.7}$$

The simultaneous transformation under which the operator A and the state vector $|m\rangle$ transform as

$$A' = UAU^{\dagger}, \tag{3.8}$$

$$|m'\rangle = U|m\rangle, \tag{3.9}$$

leaves any matrix element of the operator between the original states the same as that of the transformed operator between the transformed states. Equation (3.7) is a general result valid for any operator A and any transformation U.

Now, if we have an operator A which is invariant under the transformation U, i.e. if

$$A' = UAU^{\dagger} = A \tag{3.10}$$

or

$$[A, U] = 0, \tag{3.11}$$

we get

$$\langle m'|A|n'\rangle = \langle m|A|n\rangle\ . \tag{3.12}$$

The matrix element of the operator is the same between the original and the transformed states. We will now consider the operators that are associated with the symmetry properties of the system. If U is an operator representing the transformations, then the symmetry of the system means that

$$UHU^{\dagger} = H, \tag{3.13}$$

from this $UH = HU$ or that U commutes with H, and thus if U does not explicitly depend on time, it is a constant of motion and the physical quantity associated with it is conserved. In addition, if $|E_n\rangle$ is an eigenvector of H belonging to the eigenvalue E_n, so is $U|E_n\rangle$.

We shall discuss some specific symmetries. We will try to find explicit form of U in these cases and thereby determine the constants of motion implied by these symmetries. We note that there are four groups of symmetries that are found to be of importance in physics.

1. Continuous space-time symmetries, e.g. time translation, space translation, rotation.
2. Discrete space-time symmetries, e.g. space inversion, time reversal, particle-antiparticle conjugation.
3. Permutation symmetry, e.g. Bose-Einstein and Fermi-Dirac symmetrization.
4. Internal symmetries, e.g. charge conservation, isospin, $SU(3)$.

3.1.1 Discrete Transformations

We first consider discrete symmetry transformations. When such a transformation is applied twice, one returns to the original state. Such transformations are, therefore, described by a very simple symmetry group containing just two elements, the identity I and element g

satisfying $g^2 = 1$. Examples of such transformations are the space inversion and particle antiparticle conjugation. Let U be a linear and unitary operator; then,

$$U^\dagger = U^{-1}. \tag{3.14}$$

If U is to be an observable, it will also be Hermitian. Therefore,

$$U^\dagger = U^{-1} = U \tag{3.15}$$

$$U^\dagger U = UU = U^2 = 1. \tag{3.16}$$

The eigenvalues of this operator will then be +1 and −1. Obviously, such an operator is suitable for describing the discrete transformations mentioned above. In certain circumstances, after only one application of the transformation, the original state with utmost a phase factor is obtained. We may have, for example,

$$P|i\rangle \equiv |Pi\rangle = \eta_i|i\rangle, \tag{3.17}$$

where $U = P$, the parity operator. This means that

$$\eta_i^2 = 1, \tag{3.18}$$

$$|\eta_i| = 1 \tag{3.19}$$

because

$$\langle Pi|Pi\rangle = 1.$$

Now if $\eta_i = e^{i\lambda}$, then by redefinition of the operator $P \to Pe^{i\lambda}$, η_i can be arranged to be real. We take

$$\eta_i = \pm 1. \tag{3.20}$$

We then see that (3.17) becomes an eigenvalue equation with real eigenvalues, e.g. the pion state is an eigenstate of parity with eigenvalue −1.

Transition between states

Suppose the transitions are caused by some operator T which satisfies an exact symmetry transformation described by P. That is,

$$PTP^{-1} = T. \tag{3.21}$$

Since P is an exact symmetry transformation, the states can be partially specified by the eigenvalues of P. For the amplitude of transition,

$$\langle f|T|i\rangle = \langle f|P^{-1}\, PTP^{-1}P|i\rangle \tag{3.22}$$

$$= (\eta_f\,\eta_i)\,\langle f|T|i\rangle. \tag{3.23}$$

There are two possible solutions. Either

$$\langle f|T|i\rangle = 0,$$

which means no transition, or

$$\eta_f\eta_i = 1.$$

The transitions occur between states of the same parity. That is, if P is a symmetry group of the system, and if the system is in an initial state of parity, the final state of the parity will also be the same. We can, therefore, interpret P as an observable which is conserved in the process. If, for example, P corresponds to the operation $x \to -x$, then we say that the parity is conserved. The same is also true of the charge conjugation (particle-antiparticle conjugation) parity.

We have thus established a very important connection between invariance with respect to a set of transformation and the conservation of the observable represented by the transformation operator.

3.2 Continuous Symmetry Transformations

A continuous unitary transformation can be written in the form

$$U = \exp(iaF), \tag{3.24}$$

where a is the parameter of the transformation and the operator F is called the generator of the transformation. F is Hermitian, i.e. $F = F^{\dagger}$, and so corresponds to a physical observable. Since any finite continuous transformation can be obtained as a product of many infinitesimal transformations, we can, also for an infinitesimal transformation a, define F by means of the equation

$$U = \exp(iaF) = 1 + iaF + \ldots. \tag{3.25}$$

The requirement that U commute with H implies the requirement that F commute with H and so F is a constant of motion. As in the case of the discrete transformation, here also in any transition the value of the observable corresponding to the operator in the final state will be the same as in the initial state. If there are a number of generators F_i, the general infinitesimal transformation can be written as

$$U = 1 + ia_iF_i, \tag{3.26}$$

where a_i are just numbers and are known as the parameters of the infinitesimal transformations, whereas F_i are operators and are known as the generators of the infinitesimal transformation. These transformations generally form a group and the generators are said to constitute an algebra of the group. Among the generators F_i, a certain number r may commute among themselves and so can be used to specify the states. The r is the rank of the group of transformations. For any particular of these generators say F, since

$$[T, F] = 0, \tag{3.27}$$

taking the matrix element between the final and initial states, we have

$$\langle f | TF - FT | i \rangle = 0, \tag{3.28}$$

giving

$$(f_i - f_f)\,\langle f | T | i \rangle = 0. \tag{3.29}$$

Hence either

$$f_f = f_i \tag{3.30}$$

or

$$\langle f | T | i \rangle = 0. \tag{3.31}$$

Showing that, as in the discrete transformations, $\langle F \rangle$ is conserved in any allowed transition. Some examples of the continuous transformations are as follows:

Transformation	*Parameter(s)*	*Generator(s)*
Time translation	ε	Hamiltonian
Space translation	a_x, a_y, a_z	p_x, p_y, p_z
Rotations in three dimensions	Euler angles α, β, γ	J_x, J_y, J_z

3.3 Symmetry and Degeneracy

If a quantum mechanical system has two or more stationary states to a particular energy level, the common level is said to be degenerate. The number of states with the same energy is called the *order of degeneracy*. To illustrate the concept of degeneracy, let us consider a few examples. When an electron moves in one-dimensional potential well, the states differing only in the sign of the wave vector correspond to the same energy. These states are, therefore, two-fold degenerate. For a particle moving in a three-dimensional box the order of degeneracy is 8, the quantum numbers n_x, n_y, n_z defining the states can take up different combinations. All these eight states belong to the same energy eigenvalue. The state of an electron in a one-electron atom, described by Coulomb potential, is defined by three quantum numbers n, l, and m, if we ignore spin for the time being. The energy of the electron depends only on the principal quantum number n. The order of degeneracy can be determined as follows:

For each value of l, there are $2l + 1$ states with different values of m. For states for a given value of n and for the energy E_n then, the degeneracy d is given by

$$d = \sum_{l=0}^{n-1} (2l + 1) = n^2. \tag{3.32}$$

If we now take into account the spin of the electron, the order of degeneracy is $g = 2n^2$ because of the two states of electron spin.

Degeneracy is always associated with a symmetry of the physical system. In the first example, the electron is moving in one-dimensional potential which is symmetric under the reversal of direction (parity). In the example involving three-dimensional box, the system is symmetric under the reversal of directions along the three axes. And in the example of the hydrogen like atom, the system is symmetric with respect to all directions. This is spherical symmetry (rotational symmetry). Actually, we will see later that hydrogen like atoms have symmetry greater than the rotation symmetry.

3.3.1 Symmetry Breaking and Removal of Degeneracy

We have seen that degeneracy is due to the symmetry of the physical system. It can, therefore, be removed partially or completely by destroying the symmetry of the physical system. Then the degenerate levels are split into a series of sublevels. Usually a symmetry is broken and the degeneracy is removed under the influence of an external perturbing potential. Let us consider the following example. An electron is moving in the x-direction and has energy E. If there is no external field, the backward and forward motion of the electron are physically equivalent in the sense that both correspond to the same energy. Let us apply an electric field E along the x-axis. Then the energy of the electron along the +ve x-axis is $E - e\mathcal{E}x$ and along the –ve x-axis will be $E + e\mathcal{E}x$, where x is the distance travelled by the electron. As is clear, the external electric field $\mathcal{E}$ has destroyed the symmetry of the system and removed degeneracy. In a hydrogen-like atom the energy of an electron is given by the principal quantum number n. The states are degenerate and the order of degeneracy is $2n^2$. This degeneracy can be removed partly or wholly internally due to the spin-orbit interaction or by applying an external field.

3.4 Conservation Laws

3.4.1 Exact Conservation Laws

It is generally assumed that all interactions obey the following conservation laws:

1. Conservation of four momenta.
2. Conservation of angular momentum.
3. Conservation of electric charge.
4. Conservation of lepton number (electron number, muon number and tau number are separately conserved).
5. Conservation of baryon number.
6. Conservation of colour.
7. Invariance under CPT.

3.4.2 Quantum Numbers and Conservation Laws

Electric charge

The electric charge is one of the quantum numbers, which is exactly conserved. This conservation is associated with the masslessness of the photon. It is due to the conservation of charge that the electron cannot decay, for instance, via

$$e^- \to \nu_e + \nu_e + \bar{\nu}_e$$

or

$$e^- \to \nu_e + \gamma.$$

The lower bound for the lifetime of the electron against decay into neutral particles is

$$\tau_e^{(1)} > 5.3 \times 10^{21} \text{ years}$$

and for the decay $e^- \to \nu_e + \gamma$ is

$$\tau_e^{(2)} > 1.5 \times 10^{25} \text{ years}$$

Baryon number, B

No process has yet been observed in which B is not conserved. The stability of the proton is a critical test of the exact conservation of B. As a matter of fact, the law of conservation of B was introduced to forbid the decay of proton to positron and photon.

The baryon number such as electric charge is additive. By definition, mesons have $B = 0$, whereas baryons and antibaryons have $B = 1$ and $B = -1$, respectively. The Grand Unified Theories (GUTs) do predict proton decays, like $p \to e^+ + \pi^0$, $p \to \bar{\nu}_\mu + K^+$. The present experimental lower bound for these two channels is

$$\tau_p\ (p \to e^+ + \pi^0) > 5.9 \times 10^{32} \text{ years}$$

$$\tau_p\ (p \to K^+ + \nu_\mu) > 1.0 \times 10^{32} \text{ years}$$

Lepton number

We similarly postulate a quantum number for the leptons. All leptons carry lepton number $L = 1$, whereas all the antileptons have $L = -1$. Photons and hadrons do not carry any lepton number. We find from experiments that there are three kinds of lepton numbers. The electron and its neutrino have an electron lepton number $L_e = 1$. The muon and its neutrino have muon lepton number $L_\mu = 1$ and the τ lepton and its neutrino have tau lepton number $L_\tau = 1$. Leptons are, therefore, split into three families (or generations) with each family number conserved in all processes. For example, the muon decay follows through

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

3.4.3 Additional Conservation Laws

Additional conservation laws are obeyed by some but not all interactions.

Additional conservation laws for the strong interactions are: Parity, P; Charge conjugation, C; Isospin, I; G-parity, G; the flavour additive quantum numbers: strangeness, S; Hypercharge, Y; Charm, c; Beauty, b; and Truth, t. The status of conservation laws in all fundamental interactions is given in Table 3.1.

Table 3.1 Status of Conservation Laws in Fundamental Interactions

Conservation laws	*Strong*	*Electromagnetic*	*Weak*
Energy, Momentum, Angular momentum	yes	yes	yes
Charge Q, Baryon no. B	yes	yes	yes
Lepton numbers L_e, L_y, L_τ	yes	yes	yes
Parity P, Charge conjugation C	yes	yes	no
Strangeness S, Charm c	yes	yes	no
Isospin I	yes	no	no
CP	yes	yes	no[1]
Time reversal T	yes	yes	no[2]
CPT	yes	yes	yes

1. CP violation observed only in K^0 system.
2. Due to CP violation and CPT conservation.

3.5 Parity, Charge Conjugation and Time Reversal

3.5.1 Space Inversion (Parity)

Until 1956 it was thought that invariance under space inversion was an exact symmetry. However, the fundamental work of T.D. Lee and C.N. Yang in 1956 and subsequent experiments demonstrated that this symmetry was violated by the weak interactions.

The space inversion is a classical concept defined by the coordinate transformation

$$(x_0, \mathbf{x}) \to (x'_0, \mathbf{x}') = (x_0, -\mathbf{x}) \tag{3.33}$$

In ordinary quantum mechanics, space inversion transformation leads to the possibility of classifying states of a physical system according to a parity quantum number with value ±1. The transformation is implemented by a unitary operator P such that the coordinate, momentum, and spin operators transform according to

$$\left.\begin{aligned} \mathbf{x} \to \mathbf{x}' &= P\mathbf{x}P^{-1} = -\mathbf{x}, \\ \mathbf{p} \to \mathbf{p}' &= P\mathbf{p}P^{-1} = -\mathbf{p}, \\ \mathbf{J} \to \mathbf{J}' &= P\mathbf{J}P^{-1} = \mathbf{J}. \end{aligned}\right\} \tag{3.34}$$

whereas the state vector with definite parity satisfies

$$P\,|\,a\rangle = \pm\,|\,a\rangle. \tag{3.35}$$

Equation (3.34) keeps the commutation relations and the equations of motion unchanged.

Orbital angular momentum

When a particle state is an eigenstate of the orbital angular momentum, it is described by the spherical harmonics $Y_{l,\,m}$. In polar coordinates, the parity transformation is

$$r \to r, \qquad \theta \to \theta + \pi, \qquad \phi \to \phi - \pi\,. \tag{3.36}$$

Now,

$$Y_{lm}(\theta + \pi, \phi - \pi) = (-1)^l\, Y_{lm}. \tag{3.37}$$

Accordingly, even angular momentum states have even (+) parity and the odd angular momentum states have odd (–) parity.

Intrinsic parity

When we include the possibility of creating or destroying particles, as in relativistic quantum mechanics, we assign to each particle a definite intrinsic parity. The relative intrinsic parity of a fermion-antifermion pair is negative. For example, the particles

$$p, n, \Lambda, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^-, \Omega^-$$

all have positive intrinsic parity, whereas the corresponding antiparticles

$$\bar{p}, \bar{n}, \overline{\Lambda}, \overline{\Sigma}^+, \overline{\Sigma}^0, \overline{\Sigma}^-, \overline{\Xi}^0, \overline{\Xi}^-, \overline{\Omega}^-$$

all have negative intrinsic parity. For bosons, the corresponding antiparticle has the same parity, so that π^- and π^+ have the same intrinsic parity of (–1).

3.5.2 Parity of π^-

Parity is determined by allowing π^- capture in deuterium. This is done by slowing down the beam enough so that the π's stop in a liquid deuterium target. A π-mesic atom is formed in an $s(l = 0)$ state. The observed reactions are

$$\pi^- d \to nn, \tag{3.38a}$$

$$\to nn\gamma, \tag{3.38b}$$

but not

$$\pi^- d \to nn\pi^0. \tag{3.38c}$$

From the conservation of the angular momentum and parity, the intrinsic parity of charged pion can be determined (Problem 1) and it turns out to be odd.

π^0 parity

The non-observance of

$$\pi^- d \to nn\pi^0 \tag{3.39}$$

is suggestive of the odd parity of pion. Clearer evidence for the pseduoscalar nature of the π^0 is provided by the electromagnetic decay of π^0 to two photons, which does conserve parity. Since the spin of neutral pion is zero as shown by the argument due to L.D. Landau and C.N. Yang independently, the possible forms of the decay amplitude and so for the wavefunction of the two photons in the rest frame of the pion are

(i) $$\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_2, \tag{3.40a}$$

(ii) $$\mathbf{k} \cdot \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2, \tag{3.40b}$$

where $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are the polarization vectors of the two photons and $\mathbf{k}$ is the momentum of one of them in the rest frame of the decaying pion. Now (i) remains unchanged under parity, while (ii) changes sign. That is to say, the parity of the state is +1 for the first and –1 for the second possibility. Now (i) means that the photon polarizations are parallel, and (ii) implies that they are perpendicular so that the two possibilities can in principle be distinguished. If parity is conserved in the decay, only one of the two forms is allowed and by measuring the relative orientations of the two polarizations, we can determine whether the neutral pion is a scalar (even parity) or a pseudoscalar (odd parity) particle. However, the polarization of these photons is difficult to determine directly. Since the pair production cross-section depends on the photon polarization, the polarization information is translated into information about the relative orientation of the two production planes. The experiment favours the perpendicular orientation and so the form (ii). So the 2γ state as also π^0 has negative parity.

3.5.3 Nonconservation of Parity

There is so far no experimental evidence for parity violation in strong and electromagnetic processes. In weak processes, however, it was found that they violate this invariance principle strongly. In Chapter 6, we will see that in the so-called $V - A$ theory of the weak interactions, the parity violation turns out to be maximal as implied by the fact that vector and axial vector parts are of equal strength. This theory seems to account for all experimental data in weak processes and parity violating effects have been established in a wide variety of weak processes.

We briefly consider the situation that led to the discovery of the non-conservation of parity. In 1956, there appeared to be two kinds of K-mesons that were identical in mass, charge, and life time, but different in their decay products:

$$\theta^+ \to \pi^+ + \pi^0, \qquad (3.41a)$$

$$\tau^+ \to \pi^+ + \pi^+ + \pi^-. \qquad (3.41b)$$

If parity is assumed to be conserved in the decays, R.H. Dalitz in 1953–54 showed that θ and τ must have opposite intrinsic parities (Problem 3). These difficulties led Lee and Yang in 1956 to suggest that parity might not be conserved in the decays of K-mesons. If parity is not conserved, the same particle can decay into two pions and three pions. They suggested experiments that would test their suggestion, that parity may not be conserved in certain processes, and the experiments were performed immediately. It was found that parity is indeed not conserved in β-decay[1] and in the decay of π and μ-mesons.[2] The principle was to look for pseudoscalar terms in the weak interactions along with the scalar ones. For instance, in the β-decay of a polarized nucleus we could look for a pseudoscalar term $\mathbf{J}\,.\,\mathbf{p}$, where $\mathbf{J}$ is the spin of the nucleus and $\mathbf{P}$ the emitted electron momentum (under parity $\mathbf{P} \to -\mathbf{P}$ and $\mathbf{J} \to \mathbf{J}$). This term would produce a correlation between the direction of polarization of the nucleus and the direction of momentum of the electron; the presence of this effect would be a clear evidence of the parity violation. Wu et al. performed the experiment on the beta decay of Co^{60}

$$\mathrm{Co}^{60} \to \mathrm{Ni}^{60} + e^- + \bar{\nu}_e. \qquad (3.42)$$

What was measured was the angular distribution of the electrons emitted from polarized Co^{60} nuclei. The angular distribution is given by

$$4\pi I(\theta) = 1 + A\frac{J_z}{|J|}\frac{v}{c}\cos\theta, \qquad (3.42a)$$

where v/c is the helicity of electron, $\boldsymbol{\sigma}\cdot\mathbf{p}/|p|$, and $\mathbf{J}$ is the spin of the Co^{60} nucleus. They observed a preferential direction of emission of electrons with $A = -1$, showing the maximal violation of parity.

3.5.4 Charge Conjugation

The idea of charge conjugation first arose from the symmetry of the Dirac theory of electrons and positrons interacting with the electromagnetic field; but now it has a wider meaning due to the introduction of several other additive quantum numbers and so should perhaps be called particle-antiparticle conjugation. For historical reasons the name "charge conjugation" is still retained.

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[1] Wu, C.S. et al. (1957): *Phy. Rev.*, **105**, 1413.

[2] Garwin, R.L. et al. [8] (1957): *Phy. Rev.*, **105**, 1415.

We may remark here that invariance under charge conjugation C, like parity, holds for strong and electromagnetic interactions, while the weak interactions violate it. It seems that the appropriate particle-antiparticle transformation is the combined operation CP.

As C changes the sign of Q, L, B, S, c, b, t, we have

$$CQC = -Q, \quad \text{i.e.} \quad \{C, Q\} = 0; \quad CBC = -B, \quad \text{i.e.} \quad \{C, B\} = 0, \tag{3.43}$$

and hence we cannot simultaneously diagonalize C, Q, B ....

Clearly, eigenstates of C can only be those states for which all charges (additive quantum numbers) are zero. For example, only those states which have $Q = 0$, $B = 0$, $L = 0$, $S = 0$, etc. can be eigenstates of C. This is due to the fact that only if the particle is identical with its antiparticle, the single particle state can be the eigenstate of C. For example, the photon, the π^0 and η satisfy these conditions.

Under C the electric current changes sign as observed above. Since the electromagnetic interactions exhibit invariance under the transformation and the interaction is due to the coupling of the electromagnetic current to the electromagnetic field, the electromagnetic field must change sign under C. Since photon is the quantum of electromagnetic field we conclude that

$$C|\gamma\rangle = -|\gamma\rangle. \tag{3.44}$$

For a system of n photons,

$$C = (-1)^n. \tag{3.45}$$

Since $\pi^0 \to 2\gamma$ and C is conserved in the electromagnetic interactions, we have

$$C|\pi^0\rangle = +|\pi^0\rangle. \tag{3.46}$$

C-parity of particle-antiparticle pair

The C-parity of a particle-antiparticle pair may be deduced by considering them as two identical particles which may exist in two charge (isospin or any other internal symmetry) states. Then under total exchange of particle coordinates we must have the symmetry of the appropriate statistics: Exchange of (space) × (spin) × (charge) = – for Fermions and + for Bosons.

$(-1)^l \times (-1)^{s+1} \times C = -1$ for Fermions and $+1$ for Bosons.

For Fermion-antiFermion pair, therefore,

$$C = (-1)^{l+s}. \tag{3.47}$$

For Boson-antiBoson pair,

$$C = (-1)^{l+s+1}. \tag{3.48}$$

Positronium

Positronium is a hydrogen like bound state of an electron and a positron. It is very shortlived as it eventually decays into a number of photons. From (3.47) it is clear that

$$l + s + n = \text{even}, \tag{3.49}$$

where n is the number of photons the positronium state decays into.

Charged scalar or pseudoscalar field

We have seen in Chapter 2 that a complex scalar or pseudoscalar field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \tag{3.50}$$

where ϕ_1 and ϕ_2 are Hermitian field operators, describes spin 0 charged particles with electromagnetic four-vector current operator,

$$j_\mu = ie[(\phi^\dagger)\partial_\mu\phi - \partial_\mu\, \phi^\dagger(\phi)], \tag{3.51}$$

where $\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$. The field $\phi(\phi^\dagger)$ destroys (creates) particles and creates (destroys) antiparticles. The unitary operator C which interchanges the roles of particle and antiparticle is obviously such that

$$\phi_0^c \equiv C\phi_0C^{-1} = \phi_0^\dagger = \phi_0,$$

$$\phi^{\dagger c} \equiv C\phi^\dagger C^{-1} = \phi.$$

Clearly, the sign of j_μ (Eq. 3.51) will change under C. Examples of complex spin zero field are those used to describe the charged pions. By convention π^+ is taken as particle represented by ϕ and π^- is its antiparticle described by $\phi^\dagger$.

Another example is the pair of neutral K-mesons (K^0 and $\overline{K}^0$) which differ not in charge but in strangeness. Still another example we may take is the pair of charged K-mesons (K^+ and K^-) which differ both in charge as well as strangeness.

Neutral (Hermitian) scalar or pseudoscalar field

A neutral particle, such as the neutral pion, has no distinguishing quantum numbers like charge, strangeness, baryon number, and is described by a Hermitian field, ϕ_0. Under charge conjugation operation, ϕ_0 transforms into itself:

$$\phi_0^c \equiv C\phi_0C^{-1} = \pm\phi_0.$$

For the pion, the plus sign is chosen.

3.5.5 Time Reversal

Time reversal requires special treatment. It is somewhat more involved than the previous transformations. It requires an antiunitary operator rather than a unitary one. Since in a symmetry operation all transition probabilities are to be preserved, it was shown by Wigner that the transformation can be brought about by a unitary or an antiunitary operator.

We will first see that a unitary operator is not appropriate to describe the time reversal transformation and then show that we need the tranforming operator to be antiunitary. Let us assume that a unitary operator U changes t to $-t$. A system will be invariant under time reversal if

$$UHU^\dagger = H$$

The system in state $|\psi\rangle$ will satisfy

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle.$$

Application of the operator on the above equation gives

$$-i\frac{\partial}{\partial t}(U|\psi\rangle) = UH|\psi\rangle = H(U|\psi\rangle)$$

since U changes t to $-t$. We see that the state $U|\psi\rangle$ does not satisfy the Schrödinger equation. On the other hand, define an operator

$$T = UK, \tag{3.52}$$

where K is an operator which carries out complex conjugate on complex numbers. Then, the application of T on the Schrödinger equation

$$T\left(i\frac{\partial}{\partial t}|\psi\rangle\right) = TH|\psi\rangle$$

gives

$$i\frac{\partial}{\partial t}T|\psi\rangle = HT|\psi\rangle$$

because U changes t to $-t$ and K changes i to $-i$ and $TH = HT$ because of time reversal invariance. Thus, $T|\psi\rangle$ satisfies the same equation as does $|\psi\rangle$. So if $|\psi\rangle$ is a solution of the Schrödinger equation so is $T|\psi\rangle$. Obviously, T has the property

$$T(c_1|\psi\rangle + c_2|\phi\rangle) = c_1^*T|\psi\rangle + c_2^*T|\phi\rangle$$

of an antilinear operator and so is called antiunitary.

Equivalently, we can see as follows: In the time reversal transformation, the coordinate q and momentum p transform as

$$q \to q \qquad \text{and} \qquad p \to -p$$

so that

$$TqT^{-1} = q, \qquad TpT^{-1} = -p, \tag{3.53}$$

and to preserve the commutation relation

$$[q, p] = i, \tag{3.54}$$

we must have

$$TiT^{-1} = -i. \tag{3.55}$$

Thus, T must be an antilinear operator. An immediate consequence of the antiunitarity of T is

$$\langle T\psi_a|T\psi_b\rangle = \langle\psi_a|\psi_b\rangle^*. \tag{3.56}$$

This is a very useful relation and leads to the law of reciprocity.

Exercise 3.1 Prove the above relation.

Solution Expand the states $|\psi_a\rangle$ and $|\psi_b\rangle$ in terms of orthonormal basis:

$$|\psi_a\rangle = \sum_n |n\rangle \langle n|\psi_a\rangle \tag{3.57}$$

$$|\psi_b\rangle = \sum_m |m\rangle \langle m|\psi_b\rangle \tag{3.58}$$

Then

$$T|\psi_a\rangle = \sum_n U|n\rangle \langle n|\psi_a\rangle^* \tag{3.59}$$

$$T|\psi_b\rangle = \sum_m U|m\rangle \langle m|\psi_b\rangle^* \tag{3.60}$$

so that

$$\langle T\psi_a|T\psi_b\rangle = \sum_n\sum_m \langle \psi_a|n\rangle^* \langle n|U^\dagger U|m\rangle \langle m|\psi_b\rangle^* \tag{3.61}$$

giving

$$\langle T\psi_a|T\psi_b\rangle = \sum_n\sum_m \langle \psi_a|n\rangle^* \delta_{nm} \langle m|\psi_b\rangle^* = \sum_n \langle \psi_a|n\rangle^* \langle n|\psi_b\rangle^* = \langle \psi_a|\psi_b\rangle^* \tag{3.62}$$

Reciprocity relation

Consider the following two processes:

$$a + b \rightarrow c + d, \tag{3.63a}$$

$$c + d \rightarrow a + b. \tag{3.63b}$$

They are related by an exchange of the initial and final states. The relation is known as the reciprocity relation. From time reversal invariance, we can deduce relations between a process and the time reversed process with all spins reversed. This relationship is called principle of semi-detailed balance and is represented by the relation

$$\sum_{\text{spins}} |S_{fi}|^2 = \sum_{\text{spins}} |S_{if}|^2.$$

Reciprocity relations have been used to test time reversal invariance.

$$\frac{d\sigma}{d\Omega}(a + b \rightarrow c + d) = \frac{(2s_c + 1)(2s_d + 1)p_f^2}{(2s_a + 1)(2s_b + 1)p_i^2}\frac{d\sigma}{d\Omega}(c + d \rightarrow a + b), \tag{3.64}$$

where both cross-sections refer to the same total energy and to the same angle of scattering in the centre of mass system. An interesting application of the above relation was to the measurement of the spin of the charged pion in 1951.

3.6 CPT-Theorem

The symmetry operations C, P, and T can be combined into one operation, which is that of reversing the signs of both space and time coordinates and changing particles into antiparticles.

There are some well-known consequences of CPT invariance:

(a) The mass of a stable particle is exactly equal to the mass of its antiparticle.
(b) The lifetimes of unstable particle and its antiparticle are equal.
(c) The magnetic moments of particle and antiparticle are equal and opposite.

The most remarkable consequence of CPT invariance is that, when a Lagrangian is not invariant under one of the operations of C, P, or T, it is necessarily not invariant under one or both combined of the other symmetry operations. Thus non-conservation of P in weak interactions implies non-conservation of C or T or both combined. Another way of looking at this is that T invariance implies CP invariance.

3.7 Permutation Symmetry: System of identical particles

Two particles are identical if their physical properties are exactly the same. No observations can then distinguish them. It is, because, it is not possible, in quantum mechanics, to follow them individually in the region where they both may be found. Such complications do not arise in classical mechanics as the particles move in separate distinguishable orbits (trajectories). Furthermore, in classical mechanics two identical particles can be made distinguishable by labelling them in such a way that there is no effect on their physical process under consideration. The indistinguishability of the identical particles is thus a purely quantum mechanical property and leads to consequences of utmost importance. As a definite example, let us consider a system of, say, two electrons. The dynamics of the system is governed by the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{3.65}$$

The dynamical properties of the system remain unchanged if the two particles are interchanged. This is reflected in the fact that H in (3.65) is invariant under such a permutation. We write this symmetry as

$$H(1, 2) = P_{12}H(1, 2)\, P_{12} = H(2, 1) = H(1, 2) \tag{3.66}$$

or

$$[P_{12}, H] = 0, \tag{3.66a}$$

where P_{12} is an operator representing interchange of 1 and 2. The wavefunction of the system denoted by $\psi(1, 2)$ satisfies the Schrödinger equation

$$i\frac{\partial\psi(1, 2)}{\partial t} = H(1, 2)\,\psi(1, 2). \tag{3.67}$$

We can also write this as

$$i\frac{\partial\psi(2, 1)}{\partial t} = H(2, 1)\,\psi(2, 1) \tag{3.68}$$

because the particles are identical. Equations (3.67) and (3.66) give

$$i\frac{\partial\psi(2,1)}{\partial t} = H(1,2)\,\psi(2,1). \tag{3.69}$$

This means that $\psi(2, 1) = P_{12}\psi(1, 2)$ is a solution of the same equation (3.68) and so corresponds to the same energy. The energy levels are thus doubly degenerate, and this degeneracy is known as the exchange degeneracy. Let us take the normalized states

$$\psi_S = \frac{1}{\sqrt{2}}\,[\psi(1, 2) + \psi(2, 1)], \tag{3.70}$$

$$\psi_A = \frac{1}{\sqrt{2}}\,[\psi(1, 2) - \psi(2, 1)]. \tag{3.71}$$

$P_{12}^2 = 1$, and so the eigenvalues of P_{12} are +1 and −1. We see that

$$P_{12}\psi_S = \psi_S, \tag{3.72a}$$

$$P_{12}\psi_A = -\,\psi_A. \tag{3.72b}$$

The operator thus removes the degeneracy, as the two wavefunctions become distinguishable with the help of P_{12}. Equation (3.66) implies that P_{12} is a constant of motion, and hence a state which is symmetric at any initial time will always be symmetric and antisymmetric state will always remain antisymmetric. This can also be seen easily from (3.69). This extends to many particles.

The symmetry or anti-symmetry of the wavefunction (or state) is a characteristic of particles. This important law of nature was discovered by Pauli and states that:

1. Systems consisting of identical particles of half-odd integer spins (i.e. spin 1/2, 3/2, ...) are described by anti-symmetric wavefunctions. Such particles are called Fermions and they obey Fermi-Dirac statistics.

2. Systems of identical particles with integer spins (spin 0, 1, 2, ...) are described by the symmetric wavefunction. Such particles are called Bosons and they obey Bose-Einstein statistics.

3.7.1 System of *N*-identical Particles

We shall now generalize our results to N-particle system. We consider a system of N-identical particles having mass m, position coordinates $\mathbf{r}_i$ and momenta $\mathbf{p}_i$ ($i = 1, 2, 3, \ldots, N$). The Schrödinger equation for the wavefunction $\psi(1, 2, 3, \ldots, N)$ which describes the dynamical state of the system is

$$i\frac{\partial\psi(1,2,3,\ldots,N)}{\partial t} = H(1,2,3,\ldots,N)\,\psi(1,2,3,\ldots,N) \tag{3.73}$$

The normalization condition is

$$\int d^3r_1\, d^3r_2\, d^3r_3 \ldots d^3r_N\,|\,\psi(1, 2, 3, \ldots, N)\,|^2 = 1. \tag{3.74}$$

The Hamiltonian must be symmetric with respect to any interchange of the variables of the particles. Thus all the exchange operators P_{ij} that permute the variables of the particles i and j commute with the Hamiltonian:

$$[P_{ij}, H] = 0. \tag{3.75}$$

As a consequence, if $\psi(1, 2, 3, ..., N)$ is an eigenfunction, with eigenvalue E, so is

$$P_{ij}\psi(1, 2, 3, ..., i, j, ..., N) = \psi(1, 2, 3, ..., j, i, ..., N). \tag{3.76}$$

Since two successive interchanges bring the particles back to their initial configuration, we have

$$P_{ij}^2 = 1, \tag{3.77}$$

and so the eigenvalues of the operator P_{ij} are ±1. There are $N!$ different permutations of the variables and so there are $N\,!$ wavefunctions corresponding to the same energy. It is worth emphasizing that except for $N = 2$, the $n\,!$ permutation operations do not commute among themselves. This is due to the fact that the operators P_{ij} and P_{ik} $(k \neq j)$ do not mutually commute. (Proof is left as an exercise.) Therefore, the eigenfunction $\psi(1, 2, 3, ..., N)$ of H are not in general eigenfunctions of all the $N!$ exchange operators P_{ij}. There are, however, two exceptional states which are eigenstates of H and the $N!$ exchange operators. These are totally symmetric states $\psi_S(1, 2, 3, ..., N)$ and the totally antisymmetric states $\psi_A(1, 2, 3, ..., N)$ satisfying

$$P\psi_S = \psi_S, \tag{3.78a}$$

$$P\psi_A = -\ \psi_A, \tag{3.78b}$$

respectively for all permutations. We also note that (3.75) implies that each P is a constant of motion and so a system of identical particles represented by a wavefunction of a given symmetry (S or A) will keep that symmetry at all times. The three-particle wavefunctions properly antisymmetrized and symmetrized respecting, have the form

$$\psi_A(1, 2, 3) = \frac{1}{\sqrt{6}}[\psi(1, 2, 3) + \psi(2, 3, 1) + \psi(3, 1, 2) - \psi(2, 1, 3) - \psi(3, 2, 1) - \psi(1, 3, 2)], \tag{3.79}$$

$$\psi_S(1, 2, 3) = \frac{1}{\sqrt{6}}[\psi(1, 2, 3) + \psi(2, 3, 1) + \psi(3, 1, 2) + \psi(2, 1, 3) + \psi(3, 2, 1) + \psi(1, 3, 2)]. \tag{3.80}$$

3.8 Isospin

If we look at the tables listing the properties of the low lying meson and baryon states and their resonances, we find groups of particle states with same spin and parity and almost the same values of mass. It is convenient to put these groups of particles in multiplets which belong to the irreducible representations of the isospin group of transformations in analogy with the states $|jm\rangle$ which are the irreducible representations of the group of rotations in three-dimensional space. Indeed, the description of isospin operators and the isospin wave functions is identical to that of spin and angular momentum, the main features of which with some details are reviewed in Appendix B for reference.

It was Heisenberg who, in 1932, because of the small neutron-proton mass difference, was led to treating p and n as two different states of one particle, the nucleon. In analogy with spin, the nucleon is assigned isospin of 1/2 and p and n belong to a doublet of isospin. This description turned out to be an important tool in particle physics, especially since isospin is conserved in strong interactions and we can use the formalism of spin angular momentum. Like in angular momentum, a state is characterized by isospin I and its projection on the 3rd axis I_3. For a given I, there are $2I + 1$ states corresponding to the values of I_3, ranging from $-I$ to I. Proton and neutron are the isospin up and isospin down states respectively of isospin 1/2 nucleon N. Experimental evidence suggests that π-mesons are grouped into a triplet of isospin. Similarly, all other observed states belong to the multiplets of isospin. Members of each multiplet have nearly equal masses, and the small mass differences can be attributed to the electromagnetic corrections. If we ignore the electromagnetic interactions, systems of nucleons and pions-indeed of all isospin multiplets—are invariant under isospin group of transformations. For example,

$$\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \tag{3.81}$$

is triplet of isospin. π_1, π_2, π_3 are related to the observed pions by

$$\pi^{\pm} = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2), \qquad \pi^0 = \pi_3. \tag{3.82}$$

For a system of several particles, isospin adds as angular momentum. For two-nucleon system, for example, we have total isospin 0 and 1.

3.8.1 Isospin Wavefunctions for a Pair of Nucleons

We can represent a nucleon as a column matrix with two components

$$N = \begin{bmatrix} p \\ n \end{bmatrix}, \tag{3.83}$$

where $p(n)$ represents the probability amplitude for the nucleon to be in the $p(n)$ state. The fundamental representations $|p\rangle$ and $|n\rangle$ are denoted by the basic vectors

$$|p\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |n\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.83a}$$

Like the spin, we introduce the isospin operators $I_1 = \frac{1}{2}\tau_1$, $I_2 = \frac{1}{2}\tau_2$ and $I_3 = \frac{1}{2}\tau_3$, with

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.84}$$

which are 2 × 2 Pauli matrices acting on the fundamental states $|p\rangle$ and $|n\rangle$.

$$\tau_3|p\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |p\rangle, \tag{3.85a}$$

$$\tau_3|n\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|n\rangle. \tag{3.85b}$$

$|p\rangle$ and $|n\rangle$ are the eigenstates of τ_3 with eigenvalues +1 and –1 respectively. The lowering and raising operators are defined as

$$\tau_{\mp} = \tau_1 \mp i\tau_2 \tag{3.86}$$

so that

$$\tau_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.87}$$

$$\tau_-|p\rangle = |n\rangle, \qquad \tau_+|n\rangle = |p\rangle, \tag{3.88a}$$

$$\tau_-|n\rangle = 0, \qquad \tau_+|p\rangle = 0. \tag{3.88b}$$

$I^2 = I_1^2 + I_2^2 + I_3^2$ acting on proton or neutron states $|p\rangle$, $|n\rangle$ gives the value $\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}$. Now consider two nucleons labelled (1) and (2), their isospin being **I**(1) and **I**(2). The total isospin operator is given by

$$\mathbf{I} = \mathbf{I}(1) + \mathbf{I}(2). \tag{3.89}$$

Its values are 0 and 1. If we write the two nucleon states as eigenstates of I^2 and I_3, we can label them as $|I, I_3\rangle$. With the help of Clebsch-Gordan coefficients we write

$$\left.\begin{aligned} |1, +1\rangle &= |p(1)p(2)\rangle, \\ |1, 0\rangle &= \frac{1}{\sqrt{2}}\,[\,|p(1)n(2)\rangle + |n(1)p(2)\rangle\,], \\ |1, -1\rangle &= |n(1)n(2)\rangle, \end{aligned}\right\} \tag{3.90}$$

and

$$|0, 0\rangle = \frac{1}{\sqrt{2}}\,[\,|p(1)n(2)\rangle - |n(1)p(2)\rangle]. \tag{3.91}$$

These are in exact analogy with the wave function describing two spin 1/2 particles.

Equation (3.90) corresponding to $I = 1$ is symmetric with respect to interchange of two particles, while (3.91) corresponding to $I = 0$ is antisymmetric. The total wave function of two nucleons will be a product of space wave function, spin wave function and isospin wave function.

3.8.2 The Extended Pauli Principle

A system of two identical fermions such as protons or neutrons must, according to the Pauli exclusion principle, be described by an antisymmetric total wavefunction, that is, one that changes sign on interchange of two particles. This puts constraints on the values of isospin. For example, for the deuteron, total spin is 1, the relative orbital angular momentum is 0 (with a small $l = 2$ admixture) and so the space as well as spin wave functions are symmetric. From the generalized Pauli principle, we conclude that $I = 0$ or deuteron is an iso-singlet.

3.8.3 Consequences of Isospin Invariance

The strong interaction Hamiltonian is invariant in the isospin space and therefore commutes with all the components of the isospin **I**. Once this symmetry is assumed, relations between various cross-sections follow. Consider

$$\left.\begin{aligned} \pi^+ + p &\to \pi^+ + p, \\ \pi^- + p &\to \pi^- + p, \\ \pi^- + p &\to \pi^0 + n. \end{aligned}\right\} \tag{3.92}$$

Let us call $|I, I_3\rangle$ the state with the indicated numbers of the total isospin of the pion-nucleon system. I will take on values 3/2 or 1/2. Using Clebsch Gordan coefficients, we find that

$$|\pi^+ p\rangle = |3/1, 3/2\rangle$$

$$\left.\begin{aligned} |\pi^- p\rangle &= \sqrt{\frac{1}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \\ |\pi^0 n\rangle &= \sqrt{\frac{2}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \end{aligned}\right\} \tag{3.93}$$

Let T be the operator the matrix elements $\langle f | T | i\rangle$ of which gives the scattering amplitudes for $i \to f$. Then

$$|\langle \pi^+ p | T | \pi^+ p\rangle|^2 \tag{3.94}$$

determines the cross-section for $(d\sigma)/(d\Omega)$ for $\pi^+ p \to \pi^+ p$ at some definite energy and at certain scattering angle. Isospin symmetry implies that T cannot connect states of different I and I_3 and T has the same matrix element T_I for scattering of any member of a given I multiplet provided the energy and scattering angle are the same. That is,

$$\langle I, I_3 | T | I', I'_3\rangle = \delta_{I,I'}\delta_{I_3,I'_3} T_I . \tag{3.95}$$

Thus

$$\left.\begin{aligned} T^+ &\equiv \langle \pi^+ p | T | \pi^+ p\rangle = T_{3/2}, \\ T^- &\equiv \langle \pi^- p | T | \pi^- p\rangle = \frac{1}{3}T_{3/2} + \frac{2}{3}T_{1/2}, \\ T^0 &\equiv \langle \pi^- p | T | \pi^0 n\rangle = \frac{\sqrt{2}}{3}T_{3/2} - \frac{\sqrt{2}}{3}T_{1/2}. \end{aligned}\right\} \tag{3.96}$$

If $T_{1/2} << T_{3/2}$, then

$$\frac{d\sigma}{d\Omega}(\pi^+ p \to \pi^+ p) : \frac{d\sigma}{d\Omega}(\pi^- p \to \pi^- p) : \frac{d\sigma}{d\Omega}(\pi^- p \to \pi^0 n) = 9:1:2. \tag{3.97}$$

This condition is satisfied near $I = 3/2$ resonance, namely, at $M = 1232$ MeV. If, on the other hand, the energy is close to $I = 1/2$ resonance, then

$$\left.\begin{aligned} \frac{d\sigma}{d\Omega}(\pi^- p \to \pi^- p) : \frac{d\sigma}{d\Omega}(\pi^- p \to \pi^0 n) = 2:1, \\ \frac{d\sigma}{d\Omega}(\pi^+ p \to \pi^+ p) = 0. \end{aligned}\right\} \tag{3.98}$$

Since the amplitudes $T_{1/2}$ and $T_{3/2}$ are complex in general, we cannot derive any exact relation among the three differential cross-sections, but inequalities can be obtained. Eliminating $T_{1/2}$ and $T_{3/2}$, we have the relation

$$T^+ = T^- + \sqrt{2}\,T^0. \tag{3.99}$$

Therefore, we have the triangular inequalities

$$||T^-| - \sqrt{2}\,|T^0||^2 \le |T^+| \le [|T^-| + \sqrt{2}\,|T^0|]^2 \tag{3.100}$$

or

$$\left|\sqrt{2}\left[\frac{d\sigma}{d\Omega}(\pi^0 n)\right]^{1/2} - \left[\frac{d\sigma}{d\Omega}(\pi^- p)\right]^{1/2}\right|^2 \le \left[\frac{d\sigma}{d\Omega}(\pi^+ p)\right]^{1/2}$$

$$\le \left|\left[\frac{d\sigma}{d\Omega}(\pi^- p)\right]^{1/2} + \sqrt{2}\left[\frac{d\sigma}{d\Omega}(\pi^0 n)\right]^{1/2}\right|^2. \tag{3.101}$$

Relations (3.97) and (3.101) turn out to be pretty well obeyed for a range of pion energy.

3.9 *G*-Parity

Strong interactions are invariant with respect to charge conjugation and isospin rotations. But C does not commute with the generators of infinitesimal isospin rotations. In particular,

$$[C, I_3] \neq 0. \tag{3.102}$$

An eigenstate of I_3, for example $\pi^{\pm}$, will not in general be an eigenstate of C. (Actually, we have seen above that under C, $I_3 \to -I_3$). Charged pions transform according to

$$C|\pi^{\pm}\rangle = |\pi^{\mp}\rangle \tag{3.103}$$

or

$$C|\pi_1\rangle = +|\pi_1\rangle, \qquad C|\pi_2\rangle = -|\pi_2\rangle, \qquad C|\pi_3\rangle = +|\pi_3\rangle. \tag{3.103a}$$

Under charge symmetry operation $I_2(\pi) = e^{i\pi I_2}$, also sometimes denoted by S_c, the transformation of the pion triplet is given by

$$I_2(\pi)|\pi_1, \pi_2, \pi_3\rangle = |-\pi_1, \pi_2, -\pi_3\rangle. \tag{3.104}$$

$I_2(\pi)$ represents a 180° rotation about the y-axis in isospin space. It is convenient to define an extended charge symmetry operation G by

$$G = CI_2(\pi) \equiv Ce^{i\pi I_2}. \tag{3.105}$$

This operator was introduced by Michel and was called G-parity by Lee and Yang. This definition refers to a special choice of the frame of reference in the isospin space. It is useful in obtaining selection rules for particles which are not eigenstates of C but are eigenstates of G. The π-system has negative G-parity. It is clear from (3.103a), (3.104) and (3.105) that

$$G|\pi_1, \pi_2, \pi_3\rangle = |-\pi_1, -\pi_2, -\pi_3\rangle \tag{3.106}$$

or

$$G|\vec{\pi}\rangle = -|\vec{\pi}\rangle, \tag{3.106a}$$

i.e. the pion triplet has odd G-parity. As with the case of the C operator G is not in general an observable since

$$[G, B] \neq 0, \qquad [G, Y] \neq 0.$$

However, if the expectation value of B, Y, ... are zero, then G is an observable and is conserved in strong interactions.

3.9.1 Determination of G-parity in General for $B = 0$, $S = 0$ System

Since the G-parity should be the same for all the components of isospin multiplet, we need to find the G-parity of the neutral member only. The eigenvalue of S_c for a neutral member is given by

$$e^{i\pi I_2}|I, I_3\rangle = (-1)^{I-I_3}|I, I_3\rangle, \tag{3.107}$$

and so

$$e^{i\pi I_2}|I, 0\rangle = (-1)^I|I, 0\rangle. \tag{3.108}$$

To see this, keep in mind that $|I, I_3\rangle$ is just $Y_I^{I_3}$ and

$$Y_l^0 \propto P_l(\cos\theta). \tag{3.109}$$

Under $S_c = e^{i\pi I_2}$

$$P_l(\cos\theta) \to (-1)^l P_l(\cos\theta) \tag{3.110}$$

which means that

$$e^{i\pi I_2}\,|I, 0\rangle = (-1)^I\,|I, I_3\rangle. \tag{3.111}$$

So if we know the eigenvalue of C, we can determine the G-parity of an isospin multiplet.

3.9.2 Applications

The G-parity of a state with n pions is $(-1)^n$. From the strong decays

$$\omega \to 3\pi \quad \text{and} \quad \rho \to 2\pi, \tag{3.112}$$

we infer that

$$G(\omega) = -1, \qquad G(\rho) = +1. \tag{3.113}$$

3.9.3 Nucleonium (Nucleon-anti-nucleon system)

There are four possible states of nucleonium: $p\bar{p}, n\bar{n}, p\bar{n}$ and $n\bar{p}$. Their linear combinations which are eigenstates of I, I_3 are the following:

$$I = 0, \qquad I_3 = 0, \qquad \frac{1}{\sqrt{2}}[p\bar{p} - n\bar{n}], \tag{3.114}$$

$$I = 1 \quad \begin{cases} I_3 = 1, & p\bar{n} \\ I_3 = 0, & \frac{1}{\sqrt{2}}[p\bar{p} + n\bar{n}] \\ I_3 = -1, & n\bar{p} \end{cases} \tag{3.115}$$

These eigenstates of I, I_3 happen to be (simultaneously) eigenstates of G as well because $B = Y = 0$ for them. Under rotation of 180° about the y-axis, we have

$$S_c|I, I_3\rangle = e^{i\pi I_2}|I, I_3\rangle = (-1)^{I-I_3}|I, I_3\rangle, \tag{3.116}$$

$$S_c\,|I, 0\rangle = (-1)^I\,|I, 0\rangle. \tag{3.117}$$

Further, for any Fermion-anti-fermion system with $B = Y = Q = 0$,

$$C = (-1)^{l+s}, \qquad I_2(\pi) = (-1)^I \tag{3.118}$$

so that

$$G = (-1)^{l+s+I} \tag{3.119}$$

for the *nucleonium*. For a given value of l, s and I, we know the G-parity of the system. Now since G is conserved in strong interactions, we can find the number of pions, the nucleonium in that state will decay into. The states of the nucleonium are represented as ${}^{2s+1}(l)_{2I+1}$.

3.9.4 G-parity in Electromagnetic (EM) Interactions

The electromagnetic (EM) interactions do not conserve isospin, so that G-parity is not conserved either. But the concept of G-parity may still be useful. As a matter of fact it is possible to derive selection rules on I and G in electromagnetic transitions. Let us look at the behaviour of the electric current for the nucleon system,

$$j_\mu = \bar{p}\gamma_\mu p = \bar{N}\gamma_\mu \frac{1}{2}(1 + \tau_3)N = \frac{1}{2}\bar{N}\gamma_\mu N + \frac{1}{2}\bar{N}\gamma_\mu\tau_3 N, = j_\mu^{(s)} + j_{\mu,3}^{(V)}. \qquad (3.120)$$

The particle symbol in (3.120) represents the Dirac field for the particle. This division (3.120) of the electric current of the nucleon into iso-scalar and iso-vector parts corresponds to the Gell-Mann-Nishijima scheme

$$Q = \frac{Y}{2} + I_3. \qquad (3.121)$$

The G-parity of j^S and j^V are respectively – and + (because $G = C(-1)^I$ and $C = -1$ for photon). If we assign to a photon $I = 1$, $I_3 = 0$, the interaction $j^V \cdot A^V$ will conserve isospin, and the corresponding photon has G-parity = + like ρ^0. Similarly, the interaction $j^S \cdot A^S$ will conserve isospin if we assign $I = 0$ to the photon. This photon has $G = -$ (like ω). Thus the photon has sometimes $I = 0$ and sometimes $I = 1$, so that isospin is not conserved in photon interactions.

Applications

Consider a reaction of the type

$$A \rightarrow B + \gamma. \qquad (3.122)$$

If the states A and B are eigenstates of G, then only one of the interactions $j^S \cdot A^S$ and $j^V \cdot A^V$ will contribute to the matrix element provided G is conserved in the process. If $G_A = -G_B$, then $j^S \cdot A^S$ only contributes, the photon takes away no isospin and we have the selection rule $|I_A - I_B| = 0$. If $G_A = G_B$, then only $j^V \cdot A^V$ contributes, the photon carries away isospin unity. So we have the selection rule, $|I_A - I_B| = \pm 1, 0$, no $0 \rightarrow 0$ transitions.

Example

$$\omega \rightarrow \pi^+ + \pi^- + \gamma, \qquad (3.123a)$$

$$\rightarrow \pi^0 + \pi^0 + \gamma \qquad (3.123b)$$

$G(\omega) = -1$ and $G(2\pi) = +1$; therefore, photon emitted is iso-scalar. $I_\omega = I_{2\pi}$. Since $I_\omega = 0$, the two pions must be in the isospin zero state. From CG coefficients

$$|I = 0, I_3 = 0\rangle = \frac{1}{\sqrt{3}}|\pi^+\pi^-\rangle - \frac{1}{\sqrt{3}}|\pi^0\pi^0\rangle + \frac{1}{\sqrt{3}}|\pi^-\pi^+\rangle. \qquad (3.124)$$

So

$$\frac{\omega \rightarrow \pi^+ + \pi^- + \gamma}{\omega \rightarrow \pi^0 + \pi^0 + \gamma} = 2. \qquad (3.125)$$

The CG coefficient for the two modes ($\pi^+\pi^-$, $\pi^0\pi^0$) is the same, but due to the identity of two π^0's in the second mode, the phase space will be half of the one in the first mode and so the factor 2 on the right-hand side of Eq. (3.125).

Exercise 3.2 Find $R = \dfrac{\Gamma(\rho^+ \to \pi^+ + \gamma)}{\Gamma(\rho^0 \to \pi^0 + \gamma)}$, given $G_\rho = +1$.

Solution For η-meson, from the observation of decay

$$\eta \to 2\gamma,$$

we deduce that $C = +1$ and $G = C(-1)^I = +1$ because $I = 0$. Now,

$$\eta \to \pi^+ + \pi^- + \pi^0$$

does not conserve G, and so we conclude that this transition is electromagnetic in origin. (The weak interaction is not responsible for the process because then the probability of its occurrence would be much smaller than observed.)

3.10 Strange Particles (Gell-Mann Nishijima Scheme)

The 1950's saw the advent of particle accelerators. Cosmic rays and these accelerators produced more massive hadrons than the nucleon and the pion. They are termed strange because they are produced copiously in strong interactions yet decay weakly. A typical production and decay sequence is

$$\pi^- + p \to K^0 + \Lambda^0, \tag{3.126}$$

$$K^0 \to \pi^+ + \pi^-, \Lambda \to \pi^- + p, \tag{3.127}$$

with life times $\tau_K = 0.89 \times 10^{-10}$ s and $\tau_\Lambda = 2.63 \times 10^{-10}$ s to be compared with the strong interaction time scale of 10^{-23} s. Here Λ^0 is a neutral strange baryon whereas K^0 is a neutral strange meson. To resolve this paradox, Gell-Mann introduced a new quantum number—strangeness. Strangeness S is conserved in strong interactions, while it may not be conserved in the weak interactions. The nucleon and the pion are assigned zero strangeness, and

$$S(\Lambda^0) = -1, \qquad S(K^0) = 1. \tag{3.128}$$

Strangeness, being an additive quantum number like the electric charge, is thus conserved in the production process. Strange particles are thus produced in pairs (associate production). However, the strong decays of K and Λ are not possible as the final products have zero strangeness and so these particles decay weakly. It is convenient to introduce a quantum number hypercharge and denoted by $Y = B + S$. B is the baryon number; zero for mesons and +1 for baryons. Since the baryon number is always conserved, Y is a good quantum number for strong interactions. All the observed particles satisfy the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{Y}{2}, \tag{3.129}$$

and are arranged in isospin multiplets all the members of which have the same values of spin, parity, baryon number, strangeness, and hypercharge. The introduction of the new quantum number, strangeness or hypercharge, necessitates the enlargement of group from the isospin $SU(2)$ to a bigger group say $SU(3)$ as there appears to be correlation of the value of isospin and with that of hypercharge for all the observed multiplets.

3.10.1 Multiplets of $SU(2) \otimes U(1)_Y$

$I = 0$ iso-singlets, for example,

$$\Lambda, \quad \text{and } \eta \quad \text{with hypercharge } Y = 0.$$

$I = 1/2$ iso-doublets, e.g.

$$\begin{bmatrix} p \\ n \end{bmatrix} \text{with } Y = 1, \quad \begin{bmatrix} K^+ \\ K^0 \end{bmatrix} \text{with } Y = 1,$$

$$\begin{bmatrix} \Xi^0 \\ \Xi^- \end{bmatrix} \text{with } Y = -1, \quad \begin{bmatrix} -K^+ \\ \bar{K}^0 \end{bmatrix} \text{with } Y = -1.$$

$I = 1$ iso-triplets, e.g.

$$\begin{bmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{bmatrix} \text{with } Y = 0, \quad \begin{bmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{bmatrix} \text{with } Y = 0.$$

and so on.

The direct product of the group $SU(2) \otimes U(1)_Y$ means that there would be no correlation of the eigenvalues of I and Y of the multiplets. But from the observed multiplets we see that doublets have $Y = 1$ or -1, whereas singlets and triplets have $Y = 0$, indicating some connection of the generators of $SU(2)$ of isospin and $U(1)$ of hypercharge, implying a larger underlying symmetry group. We shall see this connection in $SU(3)$ in the next chapter.

3.11 Application of Isospin Symmetry in EM and Weak Interactions

We have seen some of the applications of isospin symmetry considerations in strong interaction where, of course, isospin is conserved. We can use isospin symmetry even in interactions where isospin is not conserved, provided we know how the interaction transforms under the isospin transformations. Even though the isospin symmetry is broken in the electromagnetic and weak interactions, it is not broken in an arbitrary manner. If isospin is conserved in an interaction, that interaction is invariant under the isospin transformation. That is, it is scalar under rotations in isospin space. On the other hand, the interactions which do not conserve isospin transform like components of an irreducible tensor. Let us, for example, consider the electromagnetic interaction $j_\mu A^\mu$. The isospin behaviour of this interaction is governed by j_μ which has the same transformation property as that of Q, that is, the combination of isoscalar and the third component of isovector.

3.11.1 Electromagnetic Interactions

From the Gell-Mann-Nishijima formula (3.129), we immediately notice that the interaction has two parts: one which is scalar (due to Y) and is invariant, whereas the second (due to

I_3) transforms like the third component of iso-vector. The magnetic moment operator due to a current is

$$\mu = \int d^3x \mathbf{r} \times \mathbf{j} \tag{3.130}$$

so that its isospin transformation property is the same as $\mathbf{j}$. We can therefore write

$$\mu = \mu\,(I = 0) + \mu\,(I = 1, I_3 = 0). \tag{3.131}$$

The value of the magnetic moment for a particle state is the expectation value of this operator in that state. For example, defining

$$\mu_{\Sigma^+} = \langle \Sigma^+ | \mu_z | \Sigma^+ \rangle, \tag{3.132}$$

we have

$$\mu_\Sigma = \langle \Sigma^+ \,|\, \mu_z(I = 0) + \mu_z(I = 1) \,|\, \Sigma^+ \rangle. \tag{3.133}$$

Using the Wigner-Eckart theorem given in Appendix B, we get

$$\mu_{\Sigma^+} = a + \langle \Sigma^+ \,|\, I_z \,|\, \Sigma^+ \rangle = a + b. \tag{3.134}$$

In an identical fashion, we obtain

$$\mu_{\Sigma^0} = a, \qquad \mu_{\Sigma^-} = a - b. \tag{3.135}$$

Eliminating the parameters a and b, we get the sum rule

$$\mu_{\Sigma^+} + \mu_{\Sigma^-} = 2\mu_{\Sigma^0}, \tag{3.136}$$

from the $SU(2)$ isospin symmetry.

3.11.2 Weak Interactions

Let us consider an example of the so-called strangeness changing hadronic decays of Λ:

$$\Lambda \rightarrow p + \pi^-, \qquad n + \pi^0. \tag{3.137}$$

We work in the rest frame of Λ. The total angular momentum of the initial state $J_i = S_\Lambda = 1/2$. Now since the spin of proton is $1/2$, the relative orbital angular momentum in the final state is $l = 0$ or 1. The relative intrinsic parity of Λ and n is taken to be positive. The parity of the final state is given by

$$(-1)^l \eta_\pi = +\,1, \tag{3.138}$$

where η_π, the intrinsic parity of π, is odd. So that $l = 1$, the p-wave, corresponds to parity conserving, whereas $l = 0$, the s-wave, corresponds to parity violating modes. Since in weak interactions parity need not be conserved, we will have both the s- and p-waves contributing to the decay. The relativistic form of the decay amplitude is dealt with in Chapter 7.

Let us now discuss the isospin properties of the decay amplitude

$$\Delta I_z = (I_z)_f - (I_z)_i = 0 - \frac{1}{2} = -\frac{1}{2}, \tag{3.139}$$

and

$$\Delta I = 1/2,\ 3/2.$$

Gell-Mann and Rosenfeld suggested the minimal change in the isospin. According to this rule, known as the $\Delta I = 1/2$ rule, we obtain (for details see Chapter 7, non-leptonic weak interactions)

$$\frac{\Gamma(\Lambda \to p\pi^-)}{\Gamma(\Lambda \to n\pi^0)} = 2. \tag{3.140}$$

This ratio is to be compared with experimental numbers 64.2% and 35.8%, in excellent agreement with prediction based on $\Delta I = 1/2$ rule.

Exercise 3.3 $(\Delta_1^{++}, \Delta_1^+, \Delta_1^0, \Delta_1^-)$ and $(\Delta_2^{++}, \Delta_2^+, \Delta_2^0, \Delta_2^-)$ are two isospin multiplets of baryon resonances. The latter decays into the former with emission of pions by strong interaction. Find the ratio of the rates of decays

$$\Delta_2^{++} \to \Delta_1^+ + \pi^+,\ \Delta_2^+ \to \Delta_1^+ + \pi^0, \text{ and } \Delta_2^+ \to \Delta_1^0 + \pi^+.$$

Solution Since the decaying particles Δ_1^+ and Δ_1^0 belong to the same isospin multiplet Δ_1, the final state baryon resonances belong to the same multiplet Δ_2 and the emitted pions to the same isotriplet π, the matrix elements for the processes are proportional to the corresponding CG coefficients with the proportionality consant (the reduced matrix element), depending upon the nature of interaction and isospins of the involved particles. That is,

$$(\Delta_1^+\pi^+ | \Delta_2^{++}) = \text{constant}\ \langle 3/2, 1/2; 1, 1 | 3/2, 3/2\rangle = -\sqrt{2/5} = -\sqrt{6/15}$$

$$(\Delta_1^+\pi^0 | \Delta_2^+) = \text{constant}\ \langle 3/2, 1/2; 1, 0 | 3/2, 1/2\rangle = \sqrt{1/15}$$

$$(\Delta_1^0\pi^+ | \Delta_2^+) = \text{constant}\ \langle 3/2, -1/2; 1, 1 | 3/2, 1/2\rangle = -\sqrt{8/15}$$

Now since the decay rate is proportional to the square of the matrix element and the phase space factor is same for all the three processes, we have

$$\Gamma(\Delta_2^{++} \to \Delta_1^+\pi^+) : \Gamma(\Delta_2^+ \to \Delta_1^+\pi^0) : \Gamma(\Delta_2^+ \to \Delta_1^0\pi^+) = 6 : 1 : 8.$$

We shall see more applications of internal symmetries like $SU(2)$, and $SU(3)$ in all the interactions; strong, electromagnetic and weak in later chapters.

3.12 Some Applications of Lorentz Covariance

3.12.1 Electromagnetic

The matrix element of the electromagnetic current operator between two states of proton of initial momentum P and final momentum P' can, on general Lorentz covariance grounds, be written as

$$\langle P' | j^\mu | P\rangle = \bar{u}(P')\ [A\gamma^\mu + BQ^\mu + Cq^\mu + D\sigma^{\mu\nu}Q_\nu + E\sigma^{\mu\nu}q_\nu]u(P),$$

where $Q = P' + P$ and $q = P - P'$. $A, B, \ldots, E$ are functions of the only Lorentz invariant, q^2.

To write the above equation, we have used the fact that the matrix element is to be a Lorentz four vector and we have written the matrix element in terms of all the vectors that can be constructed from the available four vectors and tensors.

Using Dirac equation for the spinors $u(P)$ and $\bar{u}(P')$, we can show that the terms Q^μ and $\sigma^{\mu\nu}Q_\nu$ can be written as a linear sum of the remaining three. For example,

$$\bar{u}(P')\sigma^{\mu\nu}Q_\nu u(P) = \bar{u}(P')\frac{1}{2i}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]\,Q_\nu u(P)$$

$$= \bar{u}(P')\,\frac{1}{2i}[2g^{\mu\nu} - 2\gamma^\nu\gamma^\mu]\,Q_\nu u(P) = \bar{u}(P')\frac{1}{i}[g^{\mu\nu} - \gamma^\nu\gamma^\mu]\,Q_\nu u(P)$$

$$= \bar{u}(P')\frac{1}{i}[Q^\mu - \gamma^\nu Q_\nu\gamma^\mu]u(P) = \bar{u}(P')\frac{1}{i}[Q^\mu - \gamma\cdot(P' + P)\gamma^\mu]u(P)$$

$$= \bar{u}(P')\frac{1}{i}[Q^\mu - M\gamma^\mu - \gamma\cdot P\gamma^\mu]u(P) = \bar{u}(P')\frac{1}{i}[Q^\mu - M\gamma^\mu - 2P^\mu + \gamma^\mu\gamma\cdot P]u(P)$$

$$= \bar{u}(P')\frac{1}{i}[Q^\mu - 2P^\mu]u(P) = \bar{u}(P')\frac{1}{i}[q^\mu]u(P).$$

Similarly, for the term $\bar{u}(P')Q^\mu u(P)$, we get

$$\bar{u}(P')Q^\mu u(P) = \bar{u}(P')[2M\gamma^\mu - i\sigma^{\mu\nu}q_\nu]u(P),$$

so that only three form factors are linearly independent. We may, therefore, write

$$\langle P'|j^\mu|P\rangle = \bar{u}(P')[F_1\gamma^\mu + F_3q^\mu + F_2i\sigma^{\mu\nu}q_\nu]u(P).$$

The differential form of the charge conservation law

$$\partial_\mu j^\mu = 0$$

which in the momentum space becomes

$$q_\mu\langle P'|j^\mu|P\rangle = 0$$

because

$$\partial_\mu j^\mu = i[P_\mu, j^\mu].$$

P_μ, the four-momentum operator, gives us

$$q_\mu\bar{u}(p')[F_1\gamma^\mu + F_3q^\mu + F_2i\sigma^{\mu\nu}q_\nu]u(p) = 0$$

so that

$$q^2F_3 = 0.$$

Since q^2 is in general $\neq 0$, we have $F_3 = 0$. We, therefore, write

$$\langle P'|j^\mu|P\rangle = \bar{u}(P')[F_1(q^2)\gamma^\mu + \frac{\kappa}{2M}F_2(q^2)\,\sigma^{\mu\nu}q_\nu]u(P),$$

where κ is anomalous magnetic moment of the proton and M its mass. Then,

$$F_1(0) = F_2(0) = 1.$$

F_1 and F_2 are known as the Dirac and Pauli form factors respectively. It is more convenient to use their linear combination G_E and G_M, which describes the charge and magnetic moment distributions, respectively.

3.12.2 The Strong Decay $\rho \to \pi\,\pi$

We now consider the strong decay of a spin 1 vector meson resonance decaying into two spin zero pseudoscalar mesons. Both the vector mesons as well as the pseudoscalar mesons belong to triplets of isospin. Therefore, using the Lorentz invariance and isospin symmetry we can write for the effective lagrangian the expression

$$L_{\text{eff}} = g_{\rho\pi\pi}\,\vec{\rho}_\mu \cdot (\vec{\pi} \times \vec{\pi}).$$

The matrix element describing, say $\rho^0 \to \pi^+\pi^-$ is given by

$$M = g_{\rho\pi\pi}\ \varepsilon^\mu(\rho)\ (q_1 - q_2)_\mu,$$

where $\varepsilon^\mu(\rho)$ is the polarization vector and q_1 and q_2 are the energy momentum four vectors of the outgoing pions. We have used the relevant term from the Lagrangian, viz.

$$g_{\rho\pi\pi}\ \rho^{0\mu}(\partial_\pi \pi^- \pi^+ - \pi^- \partial_\pi \pi^+).$$

If the initial vector meson is unpolarized, the average matrix element squared is

$$\frac{1}{3}\, g_{\rho\pi\pi}^2 (q_1 - q_2)^2 = \frac{1}{3}\, g_{\rho\pi\pi}^2 (\mathbf{q}_1 - \mathbf{q}_2)^2 = \frac{4}{3}\, g_{\rho\pi\pi}^2\, |\mathbf{q}|^2$$

$-\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}$ in the CM frame. The decay rate thus turns out to be

$$\Gamma(\rho \to \pi\pi) = \frac{4\pi}{32\pi^2}\,\frac{|\mathbf{q}|}{m_\rho^2}\, g_{\rho\pi\pi}^2\, \frac{4}{3}|\mathbf{q}|^2 = \frac{g_{\rho\pi\pi}^2}{4\pi}\,\frac{2}{3}\,\frac{|\mathbf{q}|^3}{m_\rho^2}.$$

PROBLEMS

Discrete Symmetries

Parity:

1. The process

$$\pi^- + d \to n + n$$

is observed in the initial s-state. The deuteron d consists of a neutron and a proton in a $^3S_1$ state. Show that the parity of the pion relative to that of the nucleon is odd.

2. Deduce from the observed process

$$K^- + \mathrm{He}^4 \to \pi^- + {}_\Lambda\mathrm{He}^4$$

that the parity of K^- relative to that of Λ is odd.

${}_\Lambda\mathrm{He}^4$ is a hyperfragment and is a bound state of two protons, a neutron and a Λ, i.e. one of the neutrons from He^4 has been replaced by a Λ. Both He^4 and ${}_\Lambda\mathrm{He}^4$ have $J^P = 0^+$. We take $\eta_\Lambda = \eta_N = +1$.

3. If τ and θ are assumed to have spin zero, show that the application of the conservation laws of angular momentum and parity to their decay modes requires that parity of τ is odd, whereas that of θ is even.
4. The η meson is a neutral particle with negative parity. Show that it cannot decay into two π^0 if parity is a good quantum number.
5. If two neutrons could form a bound state with total angular momentum 1, then it would be a negative parity state.
6. Show that a particle that decays into two π^0 will have even integer spin.
7. The ρ^0, a member of the isospin triplet (ρ^+, ρ^0, ρ^-), decays strongly into two pions. What are the charges on the pions? What are its spin and parity?
8. Consider bound states composed of spin 1/2 particles (f), with isospin 1/2. For a $\bar{f}f$ composite, show that a bound state with angular momentum l will have the quantum numbers:

$$P = (-1)^{l+1};\ C = (-1)^{l+S};\ G = (-1)^{l+S+I},$$

where S is the spin of the composite and I is its isospin.

9. Show that positronium with spin S(0, or 1) and angular momentum l has the charge conjugation

$$C = (-1)^{l+S}.$$

Thus, the $^1S_0$ state decays into two photons and $^3S_1$ into three photons. Show that the decay of a positronium into n photons is allowed if $l + S + n$ = even.

10. If the cross-section for π^+p elastic scattering rises to a peak of 180 mb at the 1238 MeV $I = 3/2$ resonance, then what will be the size for the π^-p elastic scattering?
11. Express the amplitudes describing the following reactions in terms of amplitudes of definite isospin:

$$K^+p \to K^+p, \qquad K^+n \to K^+n, \qquad K^+n \to K^0p;$$

$$K^-p \to K^-p, \qquad K^-p \to \bar{K}^0n, \qquad K^-n \to K^-n.$$

12. The $K^*(892)$ resonance has isospin 1/2, as does the K meson. It is, in fact, made of the same quarks as the K meson, with the quarks in an $S = 1$ state. Determine the relative branching ratios:

$$\frac{B(K^{*+} \to K^+\pi^0)}{B(K^{*+} \to K^0\pi^+)}$$

13. Which of the following combinations can or cannot exist in an $I = 1$ state and why?

(i) $\pi^0\pi^0$, (ii) $\pi^+\pi^-$, (iii) $\pi^+\pi^+$, (iv) $\Sigma^0\pi^0$, (v) $\Lambda^0\pi^0$

14. The $N(1470)$ is a doublet of particles (n^*, p^*) with the same quantum numbers as the doublet of nucleons. They decay into Δ-baryon resonances and π-mesons through an isospin conserving interaction. Compute the ratio of the decay rates of

$$p^* \to \Delta^{++} + \pi^-, \qquad p^* \to \Delta^+ + \pi^0, \qquad p^* \to \Delta^0 + \pi^+.$$

4

UNITARY SYMMETRY

4.1 Unitary Symmetry in 2-Dimensions

In this section we shall develop $SU(2)$ and study its properties and representations as an introduction to higher order groups $SU(n)$. Usually isospin transformations are discussed in complete analogy with rotations in a three-dimensional real space. Just as invariance under rotations in the ordinary Euclidean space implies angular momentum conservation, the invariance under rotations in isospin space implies isospin conservation. In the group theoretical classification, this symmetry group is called O_3. But we can introduce the notion of isospin invariance in a different way as we shall now see. Consider a doublet, u and d, symbolizing the up and down states in the isospin space. As is well known, any isospin rotation can be accomplished with the rotation operator

$$U(\theta) = \exp\,(i\tau \cdot n\theta/2), \tag{4.1}$$

through its effect on the u, d doublet in analogy with the ordinary spin transformations.

$$\exp\,(i\boldsymbol{\tau} \cdot \mathbf{n}\theta/2)\begin{bmatrix} u \\ d \end{bmatrix} = \exp\,(i\sum_k \tau_k\theta_k/2)\begin{bmatrix} u \\ d \end{bmatrix}. \tag{4.2}$$

The operator U can also be written as a 2 × 2 matrix with complex elements.

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \tag{4.3}$$

where

$$\det U = \alpha\alpha^* + \beta\beta^* = 1. \tag{4.4}$$

Equation (4.3) follows immediately if we note

$$U = \exp\,(i\boldsymbol{\tau} \cdot \mathbf{n}\theta/2) = \cos\,(\theta/2) + i\boldsymbol{\tau} \cdot \boldsymbol{n} \sin\,(\theta/2) \tag{4.5}$$

which can also be written as

$$U = \begin{bmatrix} \cos\,(\theta/2) + in_3 \sin(\theta/2) & (in_1 + n_2) \sin(\theta/2) \\ (in_1 - n_2) \sin\,(\theta/2) & \cos\,(\theta/2) - in_3 \sin\,(\theta/2) \end{bmatrix} \tag{4.6}$$

This is obviously of the form (4.3) with

$$\alpha = \cos(\theta/2) + in_3 \sin(\theta/2), \qquad \beta = (in_1 + n_2)\sin(\theta/2). \tag{4.7}$$

Note that (4.4) is satisfied. The matrix U has the property

$$U_{jk} U^*_{ik} = \delta_{ji} \tag{4.8}$$

or

$$U_{jk}(U^\dagger)_{ki} = \delta_{ji} \tag{4.9}$$

i.e.

$$UU^\dagger = 1,$$

and is, therefore, said to be unitary. The constraint (4.4) amounts to det $U = 1$ (unimodular condition). We may parenthetically remark that the matrix (4.3) with condition (4.4) is not the most general 2×2 unitary matrix. The most general unitary matrix is not required to be unimodular, and can be written as

$$e^{i\gamma}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}. \tag{4.10}$$

The transformations given by 2×2 unitary unimodular matrices of the type (4.3) form a continuous group $SU(2)$. It is a three-parameter group. Since the transformations can be written in the form (4.1), the generators of the $SU(2)$ transformations are the traceless τ-matrices for the lowest representation (doublet).

4.2 Lie Algebra of *SU*(2)

It is customary to define $I_i = 1/2\tau_i$ as the generators of $SU(2)$. The I_i satisfy the commutation relations:

$$[I_i, I_j] = i\varepsilon_{ijk} I_k \tag{4.11}$$

I_1, I_2 and I_3 are the x, y, and z components of the isospin operator. The square of the isospin, I^2 defined by

$$I^2 = I_1^2 + I_2^2 + I_3^2, \tag{4.12}$$

commutes with I_3 as well as with I_1 and I_2. Further, isospin is conserved in strong interactions, and so we must have

$$[I_1, H] = 0, \qquad [I_2, H] = 0, \qquad [I_3, H] = 0, \tag{4.13}$$

Hence,

$$[I^2, H] = 0. \tag{4.13a}$$

We have, therefore, simultaneous eigenstates of H, I^2, and I_3.

4.3 Fundamental Representation

We define a two-component contravariant vector or spinor or a column with two elements (representation 2).

$$q^a = \begin{bmatrix} u \\ d \end{bmatrix}, \qquad a = 1, 2 : q^1 = u, \qquad q^2 = d \tag{4.14}$$

u, d represent up and down states in analogy with spin up and spin down states. Consider the unitary transformation

$$q \to q' = Uq, \tag{4.15}$$

$$q'^a = U^a_b q^b, \tag{4.16}$$

the unimodular condition is det $U = 1$. The transformations form a special unitary group.

4.3.1 Covariant (or Conjugate) Representation

Let ψ be a representation R of a group G. If R is called a contravariant representation, the covariant representation $\bar{R}$ is defined in such a way that $\bar{\psi}\psi$ is invariant. In the particular case of the unitary transformations, $\bar{R}$ is the conjugate representation.

If we associate particles (say quarks) with the covariant spinor, then the antiparticles (antiquarks) form a contravariant spinor (representation $\bar{2}$)

$$\bar{q} = (\bar{u}, \bar{d}). \tag{4.17}$$

A contravariant vector or a row transforms as

$$\bar{q}_a \to \bar{q}'_a = \bar{q}_b (U^\dagger)^b_a. \tag{4.18}$$

A covariant two spinor can be transformed into a contravariant two spinor and vice versa with the help of antisymmetric tensors ε^{ab} and ε_{ab} because these invariant tensors can be used to raise or lower indices. We may define a contravariant spinor $q_a^{(c)}$ associated with antiparticles through

$$q^a_{(c)} = \varepsilon^{ab} \bar{q}_b. \tag{4.19}$$

$q^a_{(c)}$ transforms as $\bar{q}_b$. For $SU(2)$, R and $\bar{R}$ are equivalent representations. For the fundamental representation, it can also be seen as follows: The conjugate of the representation

$$U = \exp\left(\frac{i}{2}\tau_i n_i \theta\right) \tag{4.20}$$

is

$$U^\dagger = \exp\left(-\frac{i}{2}\tau_i n_i \theta\right) = \exp\,[i/(2\tau_i n'_i \theta)] \tag{4.21}$$

where n' is a unit vector obtained from n by 180 rotation around the second axis in isospin space.

4.4 *SU*(2) Representations

4.4.1 Product of Representations by Means of Tensor Products

All irreducible representations of $SU(2)$ can be obtained by combining the fundamental representation a number of times. Let us now perform products of representations by means of tensor products. For example, the product 2 ⊗ 2 is a tensor

$$T^{ab} = q^a q^b. \tag{4.22}$$

This representation of dimension 4 is reducible and can be split into irreducible representations by taking symmetric and antisymmetric combinations of components, because the symmetric and antisymmetric components separately transform among themselves and so are irreducible. We write

$$T^{ab} = q^a q^b = \underset{\text{symmetric}}{1/2\{q^a q^b + q^b q^a\}} + \underset{\text{antisymmetric}}{1/2\,[q^a q^b - q^b q^a]} = T^{ab} + T^{[ab]}, \tag{4.23}$$

$$4 \text{ components} = \underset{\text{isotriplet}}{3 \text{ components}} + \underset{\text{isosinglet}}{1 \text{ component}}$$

This is analogous to

$$(I = 1/2) \otimes (I = 1/2) = (I = 1) \oplus (I = 0).$$

Diquark states with definite isospin are

$I = 1$	D_1	uu	$I_z = +1,$
$I = 1$	D_2	$\frac{1}{\sqrt{2}}(ud + du)$	$I_z = 0,$
$I = 1,$	D_3	dd	$I_z = -1,$
$I = 0$	D	$\frac{1}{\sqrt{2}}(ud - du)$	$I_z = 0.$

These two quark states all have definite permutation symmetry. The isovector states are symmetric under the interchange of the constituents, whereas the isoscalar state is antisymmetric.

$$2 \otimes \bar{2}.$$

The product $2 \otimes \bar{2}$ is the tensor

$$T^a_b = \bar{q}_b q^a. \tag{4.24}$$

$q^a \bar{q}_a = \mathrm{Tr}(T)$ is an invariant and so it corresponds to representation 1. We are, then, left with a traceless tensor

$$M^a_b = T^a_b - \frac{1}{2}\delta^a_b \mathrm{Tr}(T) = \bar{q}^a q_b - \frac{1}{2}\delta^a_b \bar{q}^c q_c, \tag{4.25}$$

with three independent components. This is a tensor corresponding to the regular representation. In $SU(2)$ this representation 3 is equivalent to the one obtained in (4.23).

In terms of quarks and antiquarks, the normalized isospin eigenstates are

$$I = 0 \qquad \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \qquad I_z = 0,$$

and

$$\begin{array}{lcc} & u\bar{d} & 1, \\ I = 1 & \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) & 0, \\ & d\bar{u} & -1. \end{array} \tag{4.26}$$

The irreducible tensor M_b^a corresponding to the regular representation can be written either as an isovector π^+, π^0, π^- or as a traceless tensor (or matrix)

$$M_b^a = \begin{bmatrix} \frac{u\bar{u} - d\bar{d}}{2} & u\bar{d} \\ d\bar{u} & -\frac{u\bar{u} - d\bar{d}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{bmatrix} \tag{4.27}$$

which can also be written as

$$M_b^a = \frac{1}{\sqrt{2}} \begin{bmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{bmatrix} = \frac{1}{\sqrt{2}}(\tau \cdot \pi) \tag{4.28}$$

The isospin invariant pion nucleon interaction Lagrangian can thus be written as

$$\mathbf{L}_{\text{int}} = g\bar{N}\frac{1}{\sqrt{2}}(\boldsymbol{\tau} \cdot \boldsymbol{\pi})N = g\bar{N}_b M_a^b N^a = g[\bar{p}, \bar{n}] \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ n \end{bmatrix},$$

which also may be written as

$$\mathbf{L}_{\text{int}} = g\left[\frac{\pi^0}{\sqrt{2}}(\bar{p}p - \bar{n}n) + \pi^+\bar{p}n + \pi^-\bar{n}p\right]. \tag{4.28a}$$

Three quark states

Baryons are supposed to be made up of three quarks. In order to determine the representations of baryons, we consider the tensors made from the product of three quarks

$$q^a \otimes q^b \otimes q^c = T^{abc} = T^{(abc)} + T^{(a,\,b)c} + T^{(a,\,b)c}, \tag{4.29}$$

corresponding to

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2.$$

Totally symmetric. Quartet: $T^{(abc)}$ corresponding to $I = 3/2$.

The four Δ resonances form $I = 3/2$-quartet. The wave functions are symmetric under the interchange of any pair of quarks. We can identify the states as

$$\left.\begin{aligned}
|3/2, 3/2\rangle &= \Delta^{++} = T^{111} = uuu,\\
|3/2, 1/2\rangle &= \Delta^{+} = \frac{1}{\sqrt{3}}(uud + udu + duu),\\
&= \frac{1}{\sqrt{3}}(T^{112} + T^{121} + T^{211}),\\
|3/2, -1/2\rangle &= \Delta^{0} = \frac{1}{\sqrt{3}}(udd + dud + ddu)\\
&= \frac{1}{\sqrt{3}}(T^{122} + T^{212} + T^{221}),\\
|3/2, -3/2\rangle &= \Delta^{-} = ddd = T^{222}.
\end{aligned}\right\} \tag{4.30}$$

Inverting the above, we obtain

$$\left.\begin{aligned}
T^{111} &= \Delta^{++}, \qquad T^{112} = T^{121} = T^{211} = \Delta^{+}/\sqrt{3},\\
T^{122} &= T^{212} = T^{221} = \Delta^{0}/\sqrt{3}, \qquad T^{222} = \Delta^{-}.
\end{aligned}\right\} \tag{4.31}$$

4.4.2 Mixed Symmetric Doublet: Nucleon Doublet

In the decomposition of the product $2 \otimes 2 \otimes 2$ doublet occurs twice. There are two distinct paths to isospinor final state of three quarks. Either final quark is added to isoscalar (antisymmetric) state of first two quarks or to the isovector (symmetric) state of quarks 1 and 2.

Antisymmetric w.r.t. first two indices. $T^{[a,\, b]c}$

$$p = \frac{1}{\sqrt{2}}(ud - du)u \equiv \frac{1}{\sqrt{2}}(udu - duu), \tag{4.32}$$

$$n = \frac{1}{\sqrt{2}}(ud - du)d \equiv \frac{1}{\sqrt{2}}(udd - dud). \tag{4.33}$$

Symmetric w.r.t. first two indices. $T^{(a,\, b)c}$

$$p = \frac{1}{\sqrt{6}}(2uud - udu - duu), \tag{4.34a}$$

$$n = -\frac{1}{\sqrt{6}}(2duu - dud - udd). \tag{4.34b}$$

The tensors $T^{[a,\, b]c}$ and $T^{(a,\, b)c}$ transform like vector (N^i) in $SU(2)$ can be easily seen with the help of invariant tensor ε_{ij}. Consider

$$N^b = \varepsilon_{ac}T^{[ab]c}. \tag{4.35}$$

Let us take $b = 1$, then $a = 2$ and $c = 1$ give us

$$N^1 = \varepsilon_{21}T^{[2,1]1} = T^{[1,2]1}.$$

Identifying N^1 with proton and normalising, we obtain

$$p = \frac{1}{\sqrt{2}}(ud - du)u. \tag{4.36a}$$

Exactly similarly, starting with $b = 2$, we obtain

$$n = \frac{1}{\sqrt{2}}(ud - du)d. \tag{4.36b}$$

Now, let us consider

$$N^b = \varepsilon_{ac}T^{(ab)c}. \tag{4.37}$$

Let us again take $b = 1$. Then we can have $a = 1$ and $c = 2$ or $a = 2$ and $c = 1$, giving us

$$N^1 = \varepsilon_{12}T^{(11)2} + \varepsilon_{21}T^{(21)1} = T^{(11)2} - T^{(21)1} = 2uud - udu - duu.$$

Normalizing, we get

$$|p\rangle = \frac{1}{\sqrt{6}}(2uud - udu - duu). \tag{4.38}$$

Similarly,

$$|n\rangle = -\frac{1}{\sqrt{6}}(2duu - dud - udd). \tag{4.39}$$

4.5 Applications

Strong $\Delta \to N\pi$ *decays*

The strong interactions are invariant under isospin rotations; so the relevant Lagrangian must be a scalar. We can write

$$\mathbf{L}_{\Delta N\pi} = g_{\Delta N\pi}\bar{N}_a M^c_b \Delta^{abd}\varepsilon_{cd} \tag{4.40}$$

which on expansion becomes

$$\mathbf{L}_{\Delta N\pi} = g_{\Delta N\pi}\left[\bar{p}\pi^+\frac{\Delta_0}{\sqrt{3}} + \bar{n}\pi^+\Delta^- - \left(\bar{p}\pi^-\Delta^{++} + \bar{n}\pi^-\frac{\Delta^+}{\sqrt{3}}\right) + \frac{2}{\sqrt{6}}(\bar{p}\pi^0\Delta^+ + \bar{n}\pi^0\Delta^0)\right]. \tag{4.41}$$

So, the amplitudes for the decays are given by

$$\left.\begin{aligned} A(\Delta^{++} \to p\pi^+) &= -\, g_{\Delta N\pi} \\ A(\Delta^{+} \to n\pi^+) &= -\,\frac{1}{\sqrt{3}}\, g_{\Delta N\pi} \\ A(\Delta^{+} \to p\pi^0) &= \frac{2}{\sqrt{6}}\, g_{\Delta N\pi} \\ A(\Delta^{0} \to n\pi^0) &= \frac{2}{\sqrt{6}}\, g_{\Delta N\pi} \\ A(\Delta^{0} \to p\pi^-) &= \frac{1}{\sqrt{3}}\, g_{\Delta N\pi} \\ A(\Delta^{-} \to n\pi^-) &= g_{\Delta N\pi}. \end{aligned}\right\} \tag{4.42}$$

4.5.1 Pion-Nucleon Scattering

We now consider the pion-nucleon scattering

$\pi^+ + p \to \pi^+ + p$ represented by amplitude f_+,
$\pi^- + p \to \pi^- + p$ represented by amplitude f_-,
$\pi^- + p \to \pi^0 + n$ represented by amplitude f_0,

in the tensor formulation.

We can denote a charge independent (isospin invariant) scattering amplitude, using tensor notation, by

$$A\overline{N}_a(M^a_b M^b_c + M^b_c M^a_b)N^c + B\overline{N}_a(M^a_b M^b_c - M^b_c M^a_b)N^c. \tag{4.43}$$

The first term can also be written as

$$N_a N^a M^c_b M^b_c$$

because

$$(M^a_b M^b_c + M^b_c M^a_b) = \delta^a_c M^d_b M^b_d.$$

The relevant amplitude expression on expansion becomes

$$\begin{aligned} &A[\bar{p}(M^1_1 M^1_1 + M^1_2 M^2_1)p] + B[\bar{p}(M^1_1 M^1_1 + M^2_1 M^1_2)p] \\ &= A[\bar{p}(\pi^+\pi^- + \pi^-\pi^+ + \pi^0\pi^0)p] + A[\bar{n}(\pi^+\pi^- + \pi^-\pi^+ + \pi^0\pi^0)n] \\ &+ B[\bar{p}(\pi^-\pi^+ - \pi^-\pi^+)p] + B\frac{1}{\sqrt{2}}[\bar{p}(\pi^-\pi^0 - \pi^0\pi^- - \pi^0\pi^- + \pi^-\pi^0)n], \end{aligned} \tag{4.44}$$

so that

$$\left.\begin{aligned} f_+ &= A + B, \\ f_- &= A - B, \\ f_0 &= -\frac{2B}{\sqrt{2}}, \end{aligned}\right\} \tag{4.45}$$

giving

$$f_+ = f_- - \sqrt{2}\, f_0, \tag{4.46}$$

which for the cross-section yields

$$2\left(\frac{d\sigma_0}{d\Omega}\right)^{1/2} + \left(\frac{d\sigma_-}{d\Omega}\right)^{1/2} - \left(\frac{d\sigma_+}{d\Omega}\right)^{1/2} \geq 0. \tag{4.47}$$

This has been checked experimentally.

4.5.2 General Representations

Given any tensor, we can use ε tensors to raise and lower indices, just as the metric tensor is used in ordinary tensor analysis. In particular, we may write any tensor in terms of a tensor with all lower indices. Let

$$T_{ijklmn\ldots}$$

be a tensor of this kind. We can split T into two parts, which are respectively antisymmetric and symmetric under the interchange of, say, the first two indices. With the aid of ε_{ij}, we may write the antisymmetric part of the tensor in terms of a tensor whose rank is lowered by 2.

$$T_{[ij]klmn\ldots} = \varepsilon_{ij} T'_{klmn\ldots}.$$

Thus, we need to consider only symmetric tensors. We have an irreducible representation of $SU(2)$ associated with every space of completely symmetric tensor of a given rank with all lower indices. We will call this representation $D^{(s)}$, or (s) *for short*. The dimension of (s) is the number of linearly independent components of the tensor. Since the tensors are completely symmetric and since the indices are allowed to assume only two values, this is the same as the number of ways objects can be divided into two sets. That is to say,

$$\dim\,(s) = 2s + 1. \tag{4.48}$$

4.6 Unitary Symmetry in 3-Dimensions

4.6.1 Unitary Group: The tensor method

Consider a vector in an n-dimensional vector space. It can be represented by a column matrix in terms of its components.

$$x = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^n \end{bmatrix}. \tag{4.49}$$

A general linear transformation will take vector x to another vector x' through an $n \times n$ matrix with $2n^2$ real parameters since each element is complex in a complex linear vector space. Such transformations will form a group called $GL(n)$. If the transformation is unitary, there are n^2 conditions and so the number of real parameters will be $2n^2 - n^2 = n^2$. The transformation can be expressed through the matrix equation

$$x \to x' = ax, \tag{4.50}$$

where a is an $n \times n$ matrix with

$$a^{\dagger}a = 1 \qquad \text{(the unitarity condition).}$$

The set of all a's, i.e. all $n \times n$ unitary matrices transforming $x \to x'$, form a continuous group, namely, the unitary group $U(n)$. This group has n^2 parameters; each one of them can take continuous values. The order of $U(n)$ is thus n^2. If the condition of unimodularity, i.e. $\det a = 1$ is also imposed, the group is called the special unitary group in n dimensions and is denoted by $SU(n)$. We may write the transforming equation in the matrix form

$$\begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \\ \vdots \\ x'^n \end{bmatrix} = \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ a_1^3 & a_2^3 & a_3^3 & \cdots & a_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^n \end{bmatrix}. \tag{4.51}$$

or equivalently as a tensor equation

$$x'^{\mu} = a_{\nu}^{\mu} x^{\nu}. \tag{4.52}$$

(It is easy to see that all the group postulates are satisfied by these transformations.)

It is convenient to investigate continuous groups like $U(n)$ or $SU(n)$ by considering only those group elements which lie close to the identity element. Any finite transformation of the group can be built up from successive applications of these infinitesimal transformations. A general transformation is

$$a_{\nu}^{\mu} = \delta_{\nu}^{\mu} + \varepsilon_{\nu}^{\mu}, \tag{4.53}$$

where ε_{ν}^{μ} are the transformation parameters with

$$|\varepsilon_{\nu}^{\mu}| \ll 1.$$

The unitarity condition

$$a_{\mu}^{\lambda *} a_{\nu}^{\lambda} = \delta_{\nu}^{\mu} \tag{4.54}$$

implies

$$\varepsilon_{\mu}^{\nu *} = -\varepsilon_{\nu}^{\mu}. \tag{4.55}$$

For the $SU(n)$ group, since $\det a = 1$, we have

$$1 + \varepsilon_1^1 + \varepsilon_2^2 + \varepsilon_3^3 + \ldots + \varepsilon_n^n = 1, \tag{4.56}$$

and so

$$\varepsilon_1^1 + \varepsilon_2^2 + \varepsilon_3^3 + \ldots + \varepsilon_n^n = 0. \tag{4.57}$$

4.6.2 Generators of Infinitesimal Transformations

A representation $U(a)$ corresponding to every element a of the group may be expanded about the unit operator (or matrix) in terms of ε_μ^ν to obtain

$$U(a) = 1 - \varepsilon_\mu^\nu A_\nu^\mu, \tag{4.58}$$

where A_ν^μ is an array of operators numbering n^2 for the group $U(n)$ and $n^2 - 1$ for $SU(n)$. These generators are the infinitesimal generators of the corresponding group. Let us take the Hermitian conjugate of the operator U. Since ε are merely complex numbers, we get

$$U^\dagger(a) = 1 - \varepsilon_\mu^{\nu *} A_\nu^{\mu\dagger} = U^{-1}(a) = U(a^{-1}) = 1 + \varepsilon_\nu^\mu A_\nu^{\mu\dagger}, \tag{4.59}$$

if U is unitary and we have used (4.55). In quantum mechanics, $U(a)$ is taken as the operator which describes the transformation on the state vector of a physical system as a consequence of the element of the transformation group under consideration. $U(a)$ is thus unitary, i.e.

$$U^\dagger(a)U(a) = 1, \tag{4.60}$$

for each element a. Substituting (4.59) and (4.58) in (4.60) and ignoring second order terms, we obtain

$$1 + \varepsilon_\nu^\mu [A_\nu^{\mu\dagger} - A_\mu^\nu] = 1$$

or

$$\varepsilon_\nu^\mu [A_\nu^{\mu\dagger} - A_\mu^\nu] = 0. \tag{4.61}$$

Since ε_ν^μ are arbitrary, (4.61) gives

$$A_\nu^{\mu\dagger} = A_\mu^\nu \tag{4.62}$$

which is the hermiticity condition on generators. Substituting this back into (4.59), we have

$$U^\dagger(a) = 1 + \varepsilon_\nu^\mu A_\mu^\nu. \tag{4.63}$$

4.7 Lie-algebra for $U(n)$, $SU(n)$

We now want to obtain the Lie-algebra, a set of commutation relations among the generators of the groups $U(n)$ and $SU(n)$. To do that, let us first see how the generators themselves transform. Let a be an infinitesimal transformation and b an arbitrary transformation of the group under consideration. Then $b^{-1}ab$ is also an element of the group. We shall see that $b^{-1}ab$ is close to the identity element. That is to say, it is an infinitesimal transformation.

$$(b^{-1}ab)_\sigma^\lambda = (b^{-1})_\sigma^\mu a_\mu^\nu b_\nu^\lambda = b_\mu^{\sigma *}[\delta_\mu^\nu + \varepsilon_\mu^\nu] b_\nu^\lambda = b_\mu^{\sigma *} b_\mu^\lambda + b_\mu^{\sigma *} \varepsilon_\mu^\nu b_\nu^\lambda = \delta_\sigma^\lambda + \bar{\varepsilon}_\sigma^\lambda, \tag{4.64}$$

where

$$\bar{\varepsilon}^{\lambda}_{\sigma} = b^{\sigma *}_{\mu} \varepsilon^{\nu}_{\mu} b^{\lambda}_{\nu} \equiv (b^{-1}\varepsilon b)^{\lambda}_{\sigma}, \tag{4.65}$$

and so

$$|\bar{\varepsilon}^{\lambda}_{\sigma}| \ll 1.$$

Since $b^{-1}ab$ is an infinitesimal transformation, we can write

$$U(b^{-1}ab) = 1 - \bar{\varepsilon}^{\mu}_{\nu} A^{\nu}_{\mu}. \tag{4.66}$$

Now due to the group property, we obtain

$$U(b^{-1}ab) = U^{-1}(b)U(a)U(b) = U^{-1}(b)[1 - \varepsilon^{\mu}_{\nu} A^{\nu}_{\mu}]U(b) = 1 - \varepsilon^{\mu}_{\nu} U^{-1}(b) A^{\nu}_{\mu} U(b). \tag{4.67}$$

From (4.66) and (4.67), we get

$$\varepsilon^{\mu}_{\nu} U^{-1}(b) A^{\nu}_{\mu} U(b) = \bar{\varepsilon}^{\mu}_{\nu} A^{\nu}_{\mu} = \bar{\varepsilon}^{\lambda}_{\sigma} A^{\sigma}_{\lambda} = b^{\sigma *}_{\mu} \varepsilon^{\nu}_{\mu} b^{\lambda}_{\nu} A^{\sigma}_{\lambda} = b^{\sigma *}_{\nu} \varepsilon^{\mu}_{\nu} b^{\lambda}_{\mu} A^{\sigma}_{\lambda}. \tag{4.68}$$

Now since ε^{μ}_{ν} are arbitrary, we obtain

$$U^{-1}(b) A^{\nu}_{\mu} U(b) = b^{\sigma *}_{\nu} A^{\sigma}_{\lambda} b^{\lambda}_{\mu} \equiv (b^{\dagger})^{\nu}_{\sigma} A^{\sigma}_{\lambda} b^{\lambda}_{\mu} = b^{\lambda}_{\mu} A^{\sigma}_{\lambda} (b^{\dagger})^{\nu}_{\sigma}. \tag{4.69}$$

Suppose b is infinitesimal, i.e.

$$b^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} + \varepsilon^{\beta}_{\alpha}, \quad (b^{\dagger})^{\beta}_{\alpha} = b^{\alpha *}_{\beta} = \delta^{\beta}_{\alpha} - \varepsilon^{\beta}_{\alpha},$$

so that we can write

$$U(b) = 1 - \varepsilon^{\alpha}_{\beta} A^{\beta}_{\alpha}, \quad U^{-1}(b) \equiv U(b^{-1}) = 1 + \varepsilon^{\alpha}_{\beta} A^{\beta}_{\alpha}.$$

We have kept in mind that $b = 1 + \varepsilon$ and

$$b^{\dagger} \equiv b^{-1} = 1 + \varepsilon^{\dagger} = 1 - \varepsilon.$$

(4.69) now becomes

$$(1 + \varepsilon^{\alpha}_{\beta} A^{\beta}_{\alpha}) A^{\nu}_{\mu} (1 - \varepsilon^{\alpha}_{\beta} A^{\beta}_{\alpha}) = (\delta^{\nu}_{\sigma} - \varepsilon^{\nu}_{\sigma}) A^{\sigma}_{\lambda} (\delta^{\lambda}_{\mu} + \varepsilon^{\lambda}_{\mu}),$$

$$\text{RHS} = A^{\nu}_{\mu} + A^{\nu}_{\lambda} \varepsilon^{\lambda}_{\mu} - A^{\sigma}_{\mu} \varepsilon^{\nu}_{\sigma} + \text{second order in } \varepsilon,$$

$$\text{LHS} = A^{\nu}_{\mu} + \varepsilon^{\alpha}_{\beta} A^{\beta}_{\alpha} A^{\nu}_{\mu} - \varepsilon^{\alpha}_{\beta} A^{\nu}_{\mu} A^{\beta}_{\alpha} + \text{second order in } \varepsilon.$$

Comparing terms on both sides up to first order, we obtain

$$\varepsilon^{\alpha}_{\beta} [A^{\beta}_{\alpha}, A^{\nu}_{\mu}] = \varepsilon^{\lambda}_{\mu} A^{\nu}_{\lambda} - \varepsilon^{\nu}_{\sigma} A^{\sigma}_{\mu}$$

$$\text{RHS} = \varepsilon^{\alpha}_{\mu} A^{\nu}_{\alpha} - \varepsilon^{\nu}_{\beta} A^{\beta}_{\mu} = \delta^{\beta}_{\mu} \varepsilon^{\alpha}_{\beta} A^{\nu}_{\alpha} - \delta^{\nu}_{\alpha} \varepsilon^{\alpha}_{\beta} A^{\beta}_{\mu} = \varepsilon^{\alpha}_{\beta} [\delta^{\beta}_{\mu} A^{\nu}_{\alpha} - \delta^{\nu}_{\alpha} A^{\beta}_{\mu}].$$

Again because $\varepsilon^{\alpha}_{\beta}$ are arbitrary, we have

$$[A^{\beta}_{\alpha}, A^{\nu}_{\mu}] = \delta^{\beta}_{\mu} A^{\nu}_{\alpha} - \delta^{\nu}_{\alpha} A^{\beta}_{\mu}. \tag{4.70}$$

This defines Lie-algebra corresponding to the group $U(n)$. An array of operators T obeying

$$[A^{\beta}_{\alpha}, T^{\nu}_{\mu}] = \delta^{\beta}_{\mu} T^{\nu}_{\alpha} - \delta^{\nu}_{\alpha} T^{\beta}_{\mu}, \tag{4.71}$$

are said to be the components of a tensor operator.

4.7.1 *SU*(*n*) Lie-algebra

We have seen above that the unimodularity condition det $a = 1$ implies that $\varepsilon^{\mu}_{\mu} = 0$. There are now $n^2 - 1$ parameters and the same number of generators. The representation $U(a)$ can be written as

$$U(a) = 1 - \varepsilon^{\mu}_{\nu}[A^{\nu}_{\mu} + f\delta^{\nu}_{\mu}]$$

because $\delta^{\nu}_{\mu}\varepsilon^{\mu}_{\nu} = \varepsilon^{\nu}_{\nu} = 0$. f is an arbitrary operator. It is most convenient to choose $f = -\frac{1}{n}A^{\lambda}_{\lambda}$ and take the traceless generators

$$B^{\nu}_{\mu} = A^{\nu}_{\mu} - \frac{1}{n}\delta^{\nu}_{\mu} A^{\lambda}_{\lambda} \tag{4.72}$$

as the infinitesimal generators of $SU(n)$. Since

$$[A^{\lambda}_{\lambda}, A^{\mu}_{\nu}] = 0,$$

B^{μ}_{ν} satisfy the same commutation relations as A^{μ}_{ν}. To summarize, the generators B^{μ}_{ν} of $SU(n)$ satisfy

$$\left.\begin{aligned} B^{\mu +}_{\nu} &= B^{\nu}_{\mu}, \\ [B^{\beta}_{\alpha}, B^{\nu}_{\mu}] &= \delta^{\beta}_{\mu} B^{\nu}_{\alpha} - \delta^{\nu}_{\alpha} B^{\beta}_{\mu}, \\ B^{\lambda}_{\lambda} &= 0. \end{aligned}\right\} \tag{4.73}$$

4.7.2 *SU*(2) Generators

Take $n = 2$. We have three generators

$$B^1_2,\ B^2_1,\ \frac{1}{2}(B^1_1 - B^2_2)$$

as

$$B^1_1 + B^2_2 = 0.$$

From the commutation relations (4.73), we see that

$$\left.\begin{aligned} [B^1_2, \frac{1}{2}(B^1_1 - B^2_2)] &= B^1_2, \\ [B^2_1, \frac{1}{2}(B^1_1 - B^2_2)] &= -B^2_1, \\ [B^2_1, B^1_2] &= B^1_1 - B^2_2. \end{aligned}\right\} \tag{4.74}$$

So if we identify B_2^1, B_1^2 and $\frac{1}{2}(B_1^1 - B_2^2)$ with I_-, I_+ and I_z respectively, we obtain

$$[I_-, I_z] = I_-, \qquad [I_+, I_z] = -I_+, \qquad [I_+, I_-] = 2I_z,$$

which are the commutation relations of the isospin operators. We know that I_+, I_- are the raising and lowering operators. That is, I_+ will take a state with eigenvalue I_z to a state with eigenvalue $I_z + 1$.

Now $I^2 = \frac{1}{2}B_\nu^\mu B_\mu^\nu$ is a scalar operator as it commutes with all the three generators B_2^1, B_1^2 and $\frac{1}{2}(B_1^1 - B_2^2)$ of $SU(2)$. Then from Schur's lemma, I^2 is a *c*–number and so it can be used to label the irreducible representations. We know that the eigenvalue of I^2 is $I(I + 1)$, with $I = 0, 1/2, 1, 3/2, \ldots$. I^2 and I^2 is a Casimir operator for the group $SU(2)$.

For a particular value of I we have an irreducible representation with $2I + 1$ basic vectors. Each of these basic vectors is distinguished through its value of I_z. The basis of the irreducible representation is thus designated as

$$| I, I_z \rangle.$$

For $U(2)$, we have four generators

$$A_2^1,\ A_1^2,\ \frac{1}{2}(A_1^1 - A_2^2),\ \frac{1}{2}(A_1^1 + A_2^2),$$

$A_2^1 = I_-$, $A_1^2 = I_+$, and $\frac{1}{2}(A_1^1 - A_2^2) = I_z$ are the isospin operators and since $\frac{1}{2}(A_1^1 + A_2^2)$ commutes with all the three isospin operators, we can identify it with hypercharge Y, because hypercharge and isospin can be simultaneously measured and so their operators commute. There are two commuting operators I_z and Y and so this group has rank 2.

4.7.3 Physical Significance of *SU*(3) Generators

For the $SU(3)$ group, there are eight generators

$$B_2^1,\ B_1^2,\ \frac{1}{2}(B_1^1 - B_2^2),\ (B_1^1 + B_2^2),\ B_3^2,\ B_2^3,\ B_3^1,\ B_1^3$$

because

$$B_1^1 + B_2^2 + B_3^3 = 0.$$

Two of these, viz.

$$\frac{1}{2}(B_1^1 - B_2^2) = I_z \text{ and } (B_1^1 + B_2^2) = Y,$$

are commuting generators. None of the remaining six commute with both I_z and Y as the rank of the group is 2. We already know the physical meaning of B_2^1, B_1^2, $\frac{1}{2}(B_1^1 - B_2^2)$ and $\frac{1}{2}(B_1^1 + B_2^2)$.

Shift operators

In order to understand the significance of the remaining four, as in the isospin, let us consider the commutation relations of these four with both I_z and Y.

Problem: Use (4.73) to prove the following commutation relations:

$$\left.\begin{aligned} &[B_3^2, I_z] = [B_3^2, \tfrac{1}{2}(B_1^1 - B_2^2)] = -\tfrac{1}{2}B_3^2, \\ &[B_3^2, Y] = [B_3^2, B_1^1 + B_2^2] = B_3^2, \end{aligned}\right\} \tag{4.75}$$

$$\left.\begin{aligned} &[B_2^3, I_z] = \tfrac{1}{2}B_2^3, \qquad [B_2^3, Y] = -B_2^3, \\ &[B_3^1, I_z] = \tfrac{1}{2}B_3^1, \qquad [B_3^1, Y] = -B_3^1, \\ &[B_1^3, I_z] = -\tfrac{1}{2}B_1^3, \qquad [B_1^3, Y] = -B_1^3. \end{aligned}\right\} \tag{4.76}$$

Let both sides of (4.75) act on $|\mu(p, q)\, I, I_z, Y\rangle$, giving

$$(B_3^2 I_z - I_z B_3^2)\,|\mu(p, q)\, I, I_z, Y\rangle = -\frac{1}{2}B_3^2\,|\mu(p, q) I, I_z, Y\rangle$$

and so

$$I_z B_3^2\,|\mu(p, q)\, I, I_z, Y\rangle = (I_z + \frac{1}{2})B_3^2\,|\mu(p, q) I, I_z, Y\rangle. \tag{4.77}$$

$\mu(p, q)$ stands for the representation of the group $SU(3)$ and has two labels p, q because the rank of the group is two and there are two Casimir operators. Similarly,

$$Y B_3^2\,|\mu(p, q)\, I, I_z, Y\rangle = (Y - 1)B_3^2\,|\mu(p, q) I, I_z, Y\rangle. \tag{4.78}$$

$B_3^2 \equiv U_-$ thus raises the value of I_z by 1/2 and lowers that of Y by 1. Similarly from (4.76), we can see that

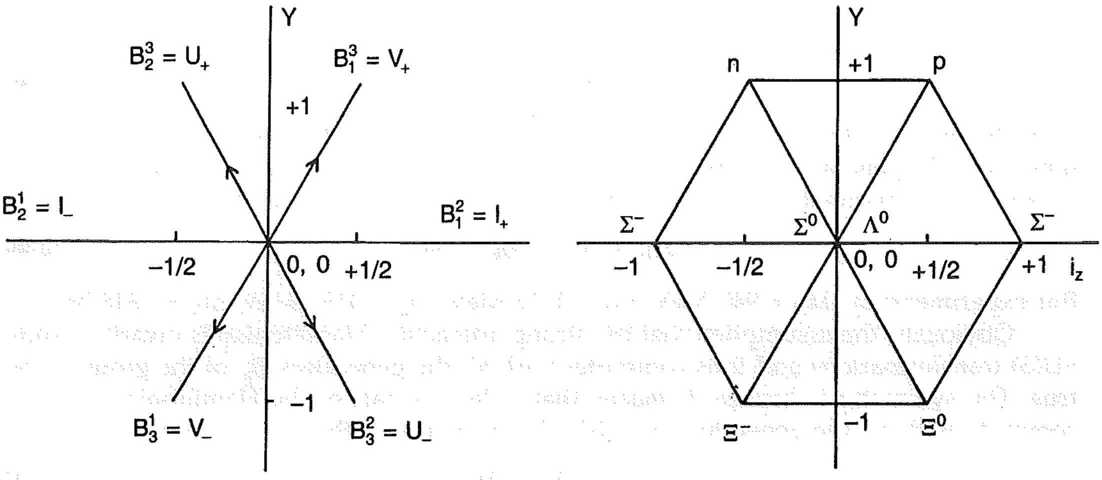
$B_2^3 \equiv U_+$ lowers the value of I_z by 1/2 and raises the value of Y by 1;

$B_3^1 \equiv V_-$ lowers the value of I_z by 1/2 and lowers the value of Y by 1;

$B_1^3 \equiv V_+$ raises the value of I_z by 1/2 and raises the value of Y by 1.

Physical significance of shift operators

With the help of shift operators (Fig. 4.1a) we can go from any basic vector (or a physical state) of an irreducible representation to any other basic vector of the irreducible representation. We have seen this in the case of isospin space where $I_\pm$ are the shift operators. In $SU(3)$ we have six such operators $I_\pm$, $V_\pm$ and $U_\pm$. If we take, for example, an octet representation (Fig. 4.1b), we can easily see either from the weight diagram or by actual operations that starting with any of the eight states, we can construct all other states with the help of these shift operators. This immediately leads us to the conclusion that if the



(a) (b)

Fig. 4.1 (a) Shift operators, and (b) octet representation.

Hamiltonian is invariant under $SU(3)$ transformations, then all the eight states are degenerate in energy. In the rest frame, then it implies that rest mass of each particle belonging to an $SU(3)$ representation is the same. All the eight baryons will have thus the same mass in $SU(3)$ symmetry limit.

We know the result from general symmetry considerations that degeneracy is a consequence of symmetry of the system (invariance of the Hamiltonian describing the system). We shall now see this result explicitly. If the Hamiltonian is invariant, we have

$$U^{-1}(a)HU(a) = H, \tag{4.79}$$

i.e.

$$[U(a), H] = 0. \tag{4.80}$$

Now,

$$U(a) = 1 - \varepsilon_\mu^\nu B_\nu^\mu .$$

So,

$$[B_\nu^\mu, H] = 0. \tag{4.81}$$

Thus in particular, all the shift operators commute with the Hamiltonian. Let us start with proton. We denote by M_p its rest mass

$$H|p\rangle = M_p|p\rangle, \tag{4.82}$$

$$B_3^2\,|p\rangle = |\Sigma^+\rangle. \tag{4.83}$$

Now because

$$[B_3^2, H] = 0, \tag{4.84}$$

$$[B_3^2, H]|p\rangle = 0, \tag{4.85}$$

$$HB_3^2\,|p\rangle = M_p B_3^2\,|p\rangle, \tag{4.86}$$

$$H|\Sigma^+\rangle = M_\Sigma|\Sigma^+\rangle = M_p|\Sigma^+\rangle. \tag{4.87}$$

So,

$$M_\Sigma = M_p. \tag{4.88}$$

With the help of other shift operators, we can show that all the eight particles have the same mass. In general, the mass of all particles belonging to an irreducible representation is same in the symmetry limit. For octet of baryons N, Λ, Σ and Ξ then, we have

$$M_N = M_\Lambda = M_\Sigma = M_\Xi. \tag{4.89}$$

But experimentally, $M_N = 940$ MeV, $M_\Lambda = 1115$ MeV, $M_\Sigma = 1193$ MeV, $M_\Xi = 1318$ MeV.

Obviously, the assumption that the strong interaction Hamiltonian is invariant under $SU(3)$ transformations and thus commutes with all the generators B^a_b of the group, is not true. The symmetry is broken. It means that at least a part of the Hamiltonian does not commute with all the generators of $SU(3)$. Let us suppose that we can write

$$H = H_0 + H_{\text{SB}}. \tag{4.90}$$

such that H_0 is invariant under $SU(3)$ transformations and H_{SB} is not. Since we can ignore the mass difference between the members of iso-spin multiplets, we shall assume isospin $SU(2)$ invariance and so H_{SB} will commute with the generators I_x, I_y, and I_z. Since hypercharge is also conserved in strong interactions, H_{SB} will commute with $Y = -B^3_3 = B^1_1 + B^2_2$ as well. If we further assume that H_{SB} transforms according to a component of octet representation (mixed tensor of rank 2), i.e. it is clear then,

$$H_{\text{SB}} = T^3_3, \tag{4.91}$$

because only component T^3_3 of the tensor operators T^a_b commutes with B^1_2, B^2_1, $\frac{1}{2}(B^1_1 - B^2_2)$ and $(B^1_1 + B^2_2)$.

T^3_3 does not commute with B^1_3, B^3_1, B^2_3 and B^3_2 and so breaks the $SU(3)$ symmetry. Using first order perturbation theory, we can write, for the mass of a particle represented by a state $|\mu(p, q)\ I, I_z, Y\rangle$, the equation

$$M\{\mu(p, q)\ I, I_z, Y\} = M_0 + \langle\mu(p, q)\ I, I_z, Y|\ T^3_3\ |\mu(p, q)\ I, I_z, Y\rangle. \tag{4.92}$$

Using the generalized ($SU(3)$) Wigner-Eckart theorem, we obtain

$$M\{\mu\ (p, q)\ I, I_z, Y\} = M_0 + bY + c[I(I + 1) - \frac{1}{4}Y^2], \tag{4.93}$$

which is the generalized Gell-Mann Okubo mass formula for the members of any irreducible representation of $SU(3)$. For the case of octet of baryons, the mass formula (4.93) becomes

$$\left.\begin{aligned} 2(M_N + M_\Xi) &= 3M_\Lambda + M_\Sigma, \\ 2(940 \text{ MeV} + 1315 \text{ MeV}) &= 3(1115 \text{ MeV}) + 1193 \text{ MeV}, \end{aligned}\right\} \tag{4.94}$$

which agrees with experimental values up to about 1%. Since, from the point of view of $SU(3)$ the octet of pseudoscalar mesons is similar to that of baryons and further $M_K = M_{\bar{K}}$ due to the CPT theorem, we have, for mesons,

$$\left.\begin{aligned} 4M_K &= 3M_\eta + M_\pi, \\ 4(495\text{ MeV}) &= 3(550\text{ MeV}) + 140\text{ MeV} \end{aligned}\right\} \tag{4.95}$$

This sum rule is not satisfied well. Instead the formula

$$\left.\begin{aligned} 4M_K^2 &= 3M_\eta^2 + M_\pi^2, \\ 4(495\text{ MeV})^2 &= 3(550\text{ MeV})^2 + (140\text{ MeV})^2 \end{aligned}\right\} \tag{4.96}$$

is in better agreement with experiment. This can be explained on the basis of field theory. In the Lagrangian density, the mass term for fermions is linear while for the mesons (bosons) it is quadratic. We, therefore, use linear mass formula for fermions and the quadratic formula for mesons.

4.8 Representations of *SU*(3)

The fundamental representations of *SU*(3) are 3 (triplet) and its conjugate $\bar{3}$ (antitriplet), see Fig. 4.2. These are identified with the quark and antiquark fields q^a and $\bar{q}_a$ respectively. $q^1 = u$, $q^2 = d$, $q^3 = s$; $\bar{q}_1 = \bar{u}$, $\bar{q}_2 = \bar{d}$ and $\bar{q}_3 = \bar{s}$.

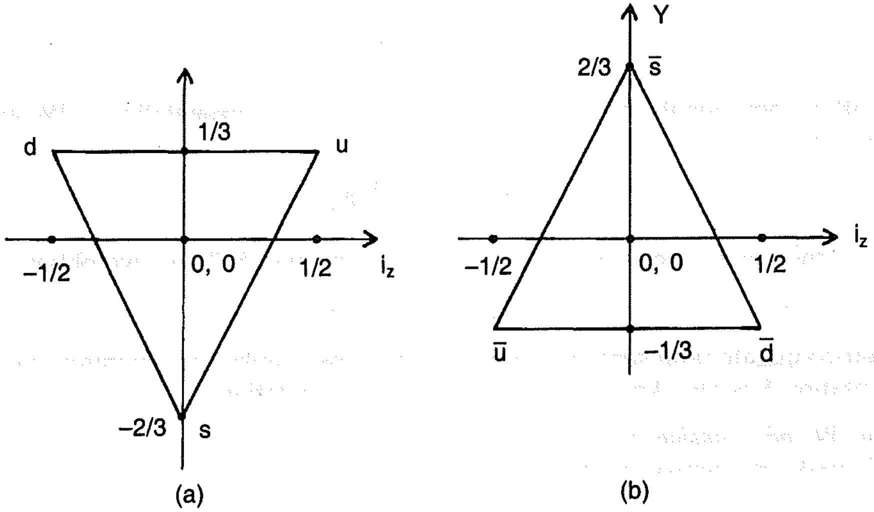


Fig. 4.2 Weight diagrams for (a) 3 (quarks), and (b) 3* (antiquarks).

4.8.1 Higher Representations

To each representation *R* (covariant), there corresponds a conjugate representation $\bar{R}$ (contravariant). In general, *R* and $\bar{R}$ are not equivalent, i.e. $R \neq \bar{R}$. If $R = \bar{R}$, the representation *R* is said to be self-conjugate.

All the representations of *SU*(3) can be built up by taking tensor products of the fundamental representations of *SU*(3). Let us denote by $q^\alpha(\alpha = 1, 2, 3)$ a contravariant vector of representation 3 and by $\bar{q}_\alpha$, a covariant vector of representation $\bar{3}$. It is easily seen that $3 \neq \bar{3}$. We can see this with the help of the invariant antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$.

In $SU(2)$, the tensor ε^{ab} and its conjugate ε_{ab} could be used to convert q^a to q_b since ε_{ab}, ε^{ab} can be used to lower or raise the index. For $SU(3)$ we cannot lower one index with the aid of $\varepsilon_{\alpha\beta\gamma}$ because $\varepsilon_{\alpha\beta\gamma} q^\gamma = q_{\alpha\beta}$ is antisymmetric with respect to α and β. A contravariant vector is thus equivalent to a covariant antisymmetric tensor of rank two.

(i) *Let us take* 3 ⊗ 3: We obtain the tensor product

$$q^{\alpha\beta} = q^\alpha q^\beta = \frac{1}{2} q^{(\alpha\beta)} + \frac{1}{2} q^{[\alpha\beta]}. \tag{4.97}$$

It has nine components. We can reduce this representation of dimension nine by taking symmetric and antisymmetric combinations as the symmetry (antisymmetry) is not affected by the $SU(3)$ transformations.

Number of components $9 = 6 \oplus 3$

As we have seen above,

$$q^{[\alpha,\beta]} = \varepsilon^{\alpha\beta\gamma} q_\gamma \tag{4.98}$$

is a representation $\bar{3}$. Therefore,

$$3 \otimes 3 = 6 \oplus \bar{3}. \tag{4.99}$$

(ii) For $3 \otimes \bar{3}$, we perform the tensor product

$$M'^{\alpha}_{\beta} = q^\alpha \bar{q}_\beta, \tag{4.100}$$

again with 9 components. $\mathrm{Tr}\,(M')$ is invariant and so of representation 1. We are left with a traceless tensor

$$M^\alpha_\beta = q^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta q^\gamma_\gamma \tag{4.101}$$

of eight components and is an irreducible representation 8. Thus, we obtain

$$3 \otimes \bar{3} = 1 \oplus 8 \tag{4.102}$$

8 is a self-conjugate representation. Since 8 is also the number of generators for $SU(3)$, this representation 8 is also known as the regular representation.

(iii) *We now consider* 6 ⊗ 3:
We make the tensor product

$$q^{(\alpha,\beta)} q^\gamma = q^{(\alpha,\beta)\gamma}.$$

We again symmetrize and antisymmetrize to obtain

$$q^{(\alpha,\beta)\gamma} = q^{(\alpha\beta\gamma)} + q^{[(\alpha,\beta),\gamma]} \tag{4.103}$$

$q^{(\alpha\beta\gamma)}$ is completely symmetric with 10 components and so is an irreducible representation of dimension 10, $q^{[(\alpha,\beta),\gamma]}$ is antisymmetric w.r.t. γ and β and symmetric w.r.t. α and β. We shall show that this tensor is equivalent to octet representation. We can now write

$$\varepsilon_{\beta\gamma\delta}\, q^{[(\alpha,\beta),\gamma]} = T^\alpha_\delta, \tag{4.104}$$

with

$$T^\delta_\delta = \varepsilon_{\beta\gamma\delta}\, q^{[(\delta,\beta),\gamma]} = 0 \tag{4.105}$$

as ε is antisymmetric and q symmetric w.r.t. δ and β. So $q^{[(\alpha,\beta),\gamma]}$ is a representation 8. We find, therefore, that

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \tag{4.106}$$

From the tensor $q^{\alpha\beta\gamma}$ we extract $q^{[\alpha\beta\gamma]}$, which is a completely antisymmetric tensor of rank 3 (representation 1) $\varepsilon^{\alpha\beta\gamma}$; completely symmetric tensor of rank 3 (representation 10) $q^{(\alpha\beta\gamma)}$, and two tensors with mixed symmetry corresponding to the representation 8.

Relations (4.102) and (4.106) are important in the formulation of the quark model. Mesons which are taken to be quark antiquark states must belong to representations 1 and 8. If baryons are made up of three quarks, they would belong to representations 1, 8, 10.

We can similarly build higher representations by treating higher rank tensors. With the aid of the completely antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$ and its conjugate $\varepsilon^{\alpha\beta\gamma}$, we can cast all irreducible tensors of $SU(3)$ into the form

$$T^{p\ \text{symmetric indices}}_{q\ \text{symmetric indices}}$$

representing a tensor with p upper symmetric indices and q lower symmetric indices. The corresponding irreducible representation is denoted by $D(p, q)$. It may be shown by simple combinatorial analysis that the dimension of $D(p, q)$ is

$$N = \frac{1}{2}(p + 1)(q + 1)(p + q + 2). \tag{4.107}$$

Self-conjugate representations are $D(p, p)$ with

$$N = (p + 1)^3,\ \text{i.e. } 1, 8, 27, 64, \ldots . \tag{4.108}$$

Two numbers p and q are needed to specify a representation. This is due to the fact that $SU(3)$ is a group of rank 2 and has, therefore, (from a theorem due to Cartan) two Casimir operators (like J^2) which commute with all the generators, and so they can be used to label the states corresponding to an irreducible representation of $SU(3)$.

4.8.2 Octet Matrices

$$M^\alpha_\beta = q^\alpha \bar{q}_\beta - \frac{1}{3}\delta^\alpha_\beta q^\gamma \bar{q}_\gamma = \begin{bmatrix} \dfrac{2u\bar{u} - d\bar{d} - s\bar{s}}{3} & u\bar{d} & u\bar{s} \\ d\bar{u} & \dfrac{2d\bar{d} - u\bar{u} - s\bar{s}}{3} & d\bar{s} \\ s\bar{u} & s\bar{d} & \dfrac{2s\bar{s} - u\bar{u} - d\bar{d}}{3} \end{bmatrix} \tag{4.109}$$

Only two diagonal elements are independent. Take

$$A = \frac{2u\bar{u} - d\bar{d} - s\bar{s}}{3}, \quad B = \frac{2d\bar{d} - s\bar{s} - u\bar{u}}{3}, \tag{4.110}$$

then

$$A - B = (u\bar{u} - d\bar{d}), \quad A + B = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{3}.$$

From isospin we know that

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) = \frac{1}{\sqrt{2}}(A - B), \tag{4.111}$$

$$\eta^0 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) = \sqrt{\frac{3}{2}}(A + B), \tag{4.112}$$

with

$$u\bar{d} = \pi^+, \quad u\bar{s} = K^+, \quad d\bar{u} = \pi^-, \quad d\bar{s} = K^0, \quad s\bar{d} = \bar{K}^0, \quad s\bar{u} = K^-.$$

We have the pseudoscalar meson matrix

$$P = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta^0 \end{bmatrix}. \tag{4.113}$$

Baryon octet

Since the $SU(3)$ quantum numbers of octet of baryons and those of octet of mesons are the same, we can write the octet of baryons also in the form of 3 × 3 matrix, namely,

$$B = \begin{bmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{bmatrix}. \tag{4.114}$$

Similarly, antibaryons can be written in the form

$$\bar{B} = \begin{bmatrix} \frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & -\frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\frac{2}{\sqrt{6}}\bar{\Lambda}^0 \end{bmatrix}. \tag{4.115}$$

because

$$(\bar{B})^{\mu}_{\nu} = \overline{B^{\nu}_{\mu}}.$$

4.8.3 Octet Baryon Wavefunction in Terms of Three Quarks

There are two different octets. One is antisymmetric under the interchange of the first two quark indices, whereas the other one is symmetric under the interchange. In the product

$$3 \otimes 3 \otimes 3 = (\bar{3} \oplus 6) \oplus 3 = 8 \oplus 1 \oplus 10 \oplus 8, \tag{4.116}$$

the first octet is obtained from $\bar{3} \otimes 3$ and the second octet from $6 \otimes 3$. In the tensor language for the first octet, we note

$$\varepsilon_{\beta\gamma\delta}\, q^{[\alpha\beta]\gamma} = B^{\alpha}_{\delta}, \tag{4.117}$$

while for the second octet observe that

$$\varepsilon_{\beta\gamma\delta}\, q^{(\alpha\beta)\gamma} = B^{\alpha}_{\delta}. \tag{4.118}$$

Octet 1

The wavefunctions of the baryons for the first octet in terms of their quark constituents are:

$$\left.\begin{aligned}
p &= \frac{1}{\sqrt{2}}(ud - du)u,\\
n &= \frac{1}{\sqrt{2}}(ud - du)d,\\
\Lambda &= \frac{1}{\sqrt{12}}[2uds - 2dus + sdu - dsu + usd - sud],\\
\Sigma^{+} &= \frac{1}{\sqrt{2}}(us - su)u,\\
\Sigma^{0} &= \frac{1}{2}[usd + dsu - sud - sdu],\\
\Sigma^{-} &= \frac{1}{\sqrt{2}}(ds - sd)d,\\
\Xi^{0} &= \frac{1}{\sqrt{2}}(us - su)s,\\
\Xi^{-} &= \frac{1}{\sqrt{2}}(ds - sd)s.
\end{aligned}\right\} \tag{4.119}$$

Octet 2

The wavefunctions of the baryons for the second octet in terms of their quark constituents are:

$$\left.\begin{aligned}
p &= \frac{1}{\sqrt{6}}(2uud - udu - duu),\\
n &= \frac{1}{\sqrt{6}}(udd + dud - 2ddu) = -\frac{1}{\sqrt{6}}(2ddu - udd - dud),\\
\Lambda &= \frac{1}{2}(usd + sud - dsu - sdu),\\
\Sigma^+ &= \frac{1}{\sqrt{6}}(2uus - usu - suu),\\
\Sigma^0 &= \frac{1}{\sqrt{12}}(2uds + 2dus - usd - dsu - sud - sdu),\\
\Sigma^- &= \frac{1}{\sqrt{6}}(2dds - dsd - sdd),\\
\Xi^0 &= \frac{1}{\sqrt{6}}(uss + sus - 2ssu),\\
\Xi^- &= \frac{1}{\sqrt{6}}(dss + sds - 2ssd).
\end{aligned}\right\} \qquad (4.120)$$

4.9 Decuplet Tensor, $D^{\alpha\beta\gamma}$

Decuplet wavefunctions constructed from the constituent quarks

$$\left.\begin{aligned}
\Delta^{++} &= D^{111} = uuu,\\
\Delta^{+} &= \frac{1}{\sqrt{3}}(D^{112} + D^{121} + D^{211}) = \frac{1}{\sqrt{3}}(uud + udu + duu),\\
\Delta^{0} &= \frac{1}{\sqrt{3}}(D^{122} + D^{212} + D^{221}) = \frac{1}{\sqrt{3}}(udd + dud + ddu),\\
\Delta^{-} &= D^{222} = ddd,\\
\Sigma^{*+} &= \frac{1}{\sqrt{3}}(D^{113} + D^{131} + D^{311}) = \frac{1}{\sqrt{3}}(uus + usu + suu),\\
\Sigma^{*0} &= \frac{1}{\sqrt{6}}(D^{123} + D^{132} + D^{213} + D^{231} + D^{312} + D^{321})\\
&= \frac{1}{\sqrt{6}}(uds + usd + dus + dsu + sud + sdu),\\
\Sigma^{*-} &= \frac{1}{\sqrt{3}}(D^{223} + D^{232} + D^{322}) = \frac{1}{\sqrt{3}}(dds + dsd + sdd),\\
\Xi^{*0} &= \frac{1}{\sqrt{3}}(D^{133} + D^{313} + D^{331}) = \frac{1}{\sqrt{3}}(uss + sus + ssu),\\
\Xi^{*-} &= \frac{1}{\sqrt{3}}(D^{233} + D^{323} + D^{332}) = \frac{1}{\sqrt{3}}(dss + sds + ssd),\\
\Omega^{-} &= D^{333} = sss.
\end{aligned}\right\} \qquad (4.121)$$

Decuplet is completely symmetric w.r.t. three indices (see Fig. 4.3).

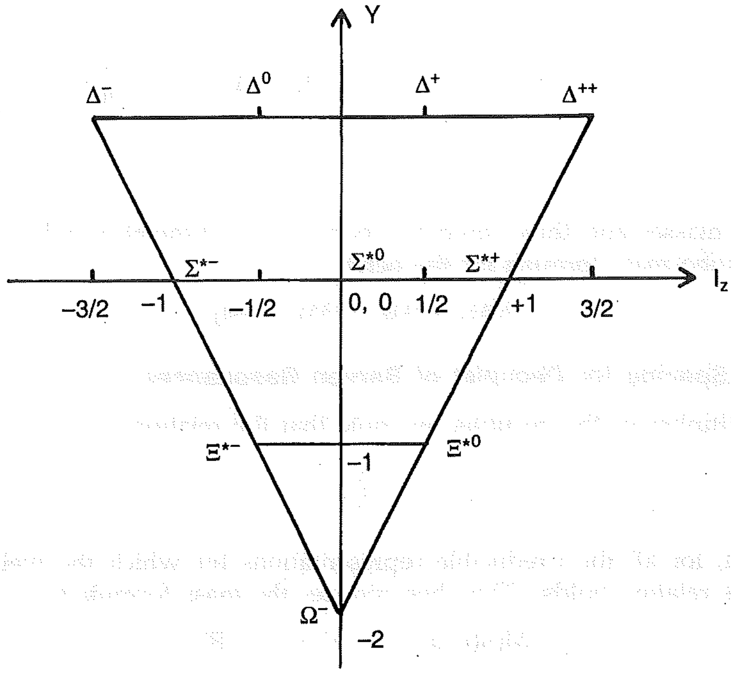


Fig. 4.3 Decuplet of resonances.

4.9.1 Mass Formula for the Octet Baryons

Tensor method

In a Lagrangian, the mass parameter appears as the coefficient of the bilinear term $\overline{\psi}\,\psi$ for a Dirac particle.

Thus, the symmetry breaking term responsible for the mass splittings of the members of an octet should be constructed from the product of two tensors B, both of which are octets. Now, if we assume that the symmetry breaking part (H_{SB}) of the interaction transforms like an eighth member (T_3^3) of an octet, we can make the product of $\overline{B}$ and B in two different ways, i.e. $\overline{B}{}_\alpha^3 B_3^\alpha$ or $\overline{B}{}_3^\alpha B_\alpha^3$. We can, therefore, write

$$\begin{aligned}
H_{\text{SB}} &= A_D(\overline{B}{}_\alpha^3 B_3^\alpha + \overline{B}{}_3^\alpha B_\alpha^3) + A_F(\overline{B}{}_\alpha^3 B_3^\alpha - \overline{B}{}_3^\alpha B_\alpha^3) \\
&= (A_D + A_F)\,\overline{B}{}_\alpha^3 B_3^\alpha + (A_D - A_F)\,\overline{B}{}_3^\alpha B_\alpha^3 \\
&= B(\overline{B}{}_1^3 B_3^1 + \overline{B}{}_2^3 B_3^2 + \overline{B}{}_3^3 B_3^3) + C(\overline{B}{}_3^1 B_1^3 + \overline{B}{}_3^2 B_2^3 + \overline{B}{}_3^3 B_3^3) \\
&= B\left(\overline{p}p + \overline{n}n + \frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\overline{\Lambda}{}^0\Lambda^0\right) + C\left(\overline{\Xi}{}^-\Xi^- + \overline{\Xi}{}^0\Xi^0 + \frac{2}{\sqrt{6}}\frac{2}{\sqrt{6}}\overline{\Lambda}{}^0\Lambda^0\right),
\end{aligned} \tag{4.122}$$

where $B = A_D + A_F$ and $C = A_D - A_F$. The contribution coming from the $SU(3)$ part of the Hamiltonian is the same for all the particles. We, therefore, have

$$\left.\begin{aligned} M_N &= A + (A_D + A_F), \\ M_\Lambda &= A + \frac{2}{3}(A_D + A_F + A_D - A_F) = A + \frac{4}{3}A_D, \\ M_\Sigma &= A, \\ M_\Xi &= A + (A_D - A_F). \end{aligned}\right\} \tag{4.123}$$

We have four masses and three unknown parameters, eliminating which we obtain the Gell-Mann Okubo mass formula for the octet.

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma. \tag{4.124}$$

4.9.2 Equal Spacing for Decuplet of Baryon Resonances

For the isomultiplets of the decuplet, we note that the relation

$$I = \frac{Y}{2} + 1 \tag{4.125}$$

holds. (In fact, for all the irreducible representations for which the weight diagram is triangular, this relation holds). This then reduces the mass formula to

$$M(\mu(p, q); I, I_z, Y) = A + BY, \tag{4.126}$$

resulting in equal spacing for the decuplet baryon resonant states as the isospin multiplets are separated by one unit of y. On the basis of this equal spacing, Gell-Mann predicted, in 1961, a new particle Ω^- with mass to be about 1680 MeV, the discovery of which in 1964 gave a big boost to the $SU(3)$ considerations. The observed spilttings are

$$\Omega^-\,(1670) - \Xi^*\,(1530) = 142\text{ MeV}, \qquad \Xi^*\,(1530) - \Sigma^*\,(1385) = 145\text{ MeV}.$$

$$\Sigma^*\,(1385) - \Delta\,(1236) = 149\text{ MeV}.$$

in agreement with predictions.

Alternative derivation (Tensor method)

Here, H_{SB} can be written as

$$H_{\text{SB}} = A_{\text{SB}}\,\overline{D}_{\alpha\beta 3}D^{\alpha\beta 3}, \tag{4.127}$$

with

$$\begin{aligned} \overline{D}_{\alpha\beta 3}D^{\alpha\beta 3} &= \overline{D}_{1\beta 3}D^{1\beta 3} + \overline{D}_{2\beta 3}D^{2\beta 3} + \overline{D}_{3\beta 3}D^{3\beta 3} \\ &= \overline{D}_{113}D^{113} + \overline{D}_{123}D^{123} + \overline{D}_{133}D^{133} + \overline{D}_{213}D^{213} + \overline{D}_{223}D^{223} \\ &\quad + \overline{D}_{233}D^{233} + \overline{D}_{313}D^{313} + \overline{D}_{323}D^{323} + \overline{D}_{333}D^{333} \\ &= \overline{D}_{113}D^{113} + 2\overline{D}_{123}D^{123} + 2\overline{D}_{133}D^{133} + \overline{D}_{223}D^{223} + 2\overline{D}_{233}D^{233} + \overline{D}_{333}D^{333} \\ &= \frac{1}{3}\Sigma^{*+}\Sigma^{+} + \frac{2}{6}\Sigma^{*0}\Sigma^{0} + \frac{2}{3}\overline{\Xi}^{*0}\Xi^{*0} + \frac{1}{3}\overline{\Sigma}^{*-}\Sigma^{-} + \frac{2}{3}\overline{\Xi}^{*-}\Xi^{*-} + \overline{\Omega}^{-}\Omega^{-} \end{aligned} \tag{4.128}$$

giving for the masses

$$\left.\begin{aligned} M_\Delta = A, \qquad M_{\Sigma^*} = A + \frac{1}{3} A_{\text{SB}}, \\ M_{\Xi^*} = A + \frac{2}{3} A_{\text{SB}}, \qquad M_\Omega = A + A_{\text{SB}}. \end{aligned}\right\} \tag{4.129}$$

4.9.3 Vector Meson Nonet

An $SU(3)$ octet of mesons contains two isospin doublets with hypercharge +1 and –1. Both these doublets have the same mass as the members of one doublet are the antiparticles of the members of the other doublet. The other four members belong to an iso-triplet and a singlet. If we denote the masses of doublet, triplet and singlet by $M_{1/2}$, M_1 and M_0, the Gell-Mann Okubo formula reads

$$4M_{1/2}^2 = M_1^2 + 3M_0^2. \tag{4.130}$$

Experimentally the nine meson states, with $J^P = 1^-$ are tabulated in Chapter 1.

We may like to know which of the two states with $I = 0$ and $Y = 0$ along with the seven states (ρ, K^*, $\overline{K}^*$) belongs to the octet of vector mesons. If we label the eighth member with $I = 0$, $Y = 0$ as ω_8, then its mass is given by the Gell-Mann Okubo mass formula and is

$$M^2(\omega_8) = \frac{4M^2(K^*) - M^2(\rho)}{3} = (928\text{ MeV})^2. \tag{4.131}$$

Neither ω nor ϕ has mass near 928 MeV. However, the predicted mass lies between the value of the two masses. We, therefore, consider the nonet of vector mesons with the possibility of a certain linear combination of ω and ϕ belonging to the octet, but the orthogonal combination being an $SU(3)$ singlet.

4.9.4 The ω-ϕ Mixing

Since the Hamiltonian of the strong interaction has a symmetry breaking part which transforms like an $SU(3)$ octet member, there can be a non-zero matrix element of this part of the Hamiltonian between the two representations 8 and 1. So the symmetry breaking can mix the (iso-singlet) eighth member of the octet with the the $SU(3)$ singlet. Let us suppose that the 8 and 1 particles have the same mass when H_{SB} is switched off. This is the so-called nonet model. To treat the effect of H_{SB}, we use the degenerate perturbation theory. H_{SB} will break the degeneracy, but there will still be isospin degeneracy as H_{SB} conserves isospin. Triplet of ρ and doublets of K^* and $\overline{K}^*$ will still have the same masses $M(\rho)$ and $M(K^*)$ respectively. Since, ω_8 and the $SU(3)$-singlet ω_1 have the same values for the quantum numbers I, I_z and Y, there will be mixing between these due to H_{SB}. So the eigenstates of the mass matrix—the physical states ω and ϕ which we observe in nature—are, in general, a linear combination of ω_8 and ω_1. The mass matrix in the octet singlet basis (ω_8 and ω_1) is given by

$$M^2 = \begin{bmatrix} M_8^2 & M_{81}^2 \\ M_{18}^2 & M_1^2 \end{bmatrix}, \tag{4.132}$$

where

$$\left.\begin{aligned} M_8 &= \langle\omega_8\,|\,(H_{\text{inv}} + H_{\text{SB}})\,|\,\omega_8\rangle, \\ M_1 &= \langle\omega_1\,|\,(H_{\text{inv}} + H_{\text{SB}})\,|\,\omega_1\rangle, \\ M_{81} = M_{18} &= \langle\omega_1\,|\,(H_{\text{inv}} + H_{\text{SB}})\,|\,\omega_8\rangle. \end{aligned}\right\} \tag{4.133}$$

We have taken

$$|\omega_8\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\omega_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.134}$$

To find the physical states we diagonalize the mass matrix. Take

$$|\,\phi\rangle = \cos\theta\,|\,\omega_8\rangle + \sin\theta\,|\,\omega_1\rangle, \tag{4.135a}$$

$$|\,\omega\rangle = -\sin\theta\,|\,\omega_8\rangle + \cos\theta\,|\,\omega_1\rangle, \tag{4.135b}$$

to ensure

$$\langle\phi\,|\,\omega\rangle = 0.$$

This gives us the mass matrix

$$M^2 = \begin{bmatrix} M_\phi^2 & 0 \\ 0 & M_\omega^2 \end{bmatrix}.$$

in the physical basis. θ is determined from the diagonalization, i.e.

$$\begin{bmatrix} M_8^2 & M_{81}^2 \\ M_{18}^2 & M_1^2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = M_\phi^2 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \tag{4.136a}$$

$$\begin{bmatrix} M_8^2 & M_{81}^2 \\ M_{18}^2 & M_1^2 \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = M_\omega^2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}. \tag{4.136b}$$

It is easy to see that

$$\tan\theta = \frac{M_\phi^2 - M_8^2}{M_{81}^2} = -\frac{M_{18}^2}{M_1^2 - M_\phi^2}, \tag{4.137}$$

$$\tan\theta = \frac{M_{81}^2}{M_8^2 - M_\omega^2} = -\frac{M_{18}^2}{M_1^2 - M_\omega^2}, \tag{4.138}$$

so that

$$\tan^2\theta = \frac{M_\phi^2 - M_8^2}{M_8^2 - M_\omega^2}. \tag{4.139}$$

With the observed values of the masses, θ is nearly equal to 35°. So ϕ is about 65% in ω_8 state and 35% in ω_1 state. We can write, approximately,

$$|\phi\rangle = \sqrt{2/3}\,|\omega_8\rangle + \sqrt{1/3}\,|\omega_1\rangle, \tag{4.140a}$$

$$|\omega\rangle = -\sqrt{1/3}\,|\omega_8\rangle + \sqrt{2/3}\,|\omega_1\rangle. \tag{4.140b}$$

This is a result of the nonet model, where in terms of the quark wave functions the observed states are given by

$$\omega = \sqrt{1/2}\,(u\bar{u} + d\bar{d}), \qquad \phi = s\bar{s}. \tag{4.141}$$

Since s quark is heavier than u and d, ϕ is heavier than ω. The value of θ can be used to predict the decay rate of

$$\phi \to K\bar{K}$$

from that of $\rho \to 2\pi$ and $K^* \to K\pi$. The experimental result is close to the prediction when θ is taken to be 35°. In the nonet model, the nine (eight + one) vector meson states are expresssed as a mixed tensor V^α_β represented by a 3 × 3 matrix

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}}(\omega + \rho^0) & \rho^+ & K^{*+} \\ \rho^- & \frac{1}{\sqrt{2}}(\omega - \rho^0) & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{bmatrix}$$

PROBLEMS

SU(3)

1. Deduce the expression for the dimensionality of a general irreducible tensor corresponding to an irreducible representation of *SU*(3).
2. By explicit use of the invariant tensors δ^i_j and ε^{ijk} and ε_{ijk} obtain expressions for the 27, the 10, the $\overline{10}$, the symmetric 8, the antisymmetric 8 and the singlet, which are the irreducible representations appearing in the tensor product of two *SU*(3) octets.
3. (a) Discuss the validity of the Gell-Mann Okubo mass formula for the vector mesons, explaining the ω-ϕ mixing.

   (b) If the physically observed ω, ϕ states are given by

$$|\omega\rangle = \cos\theta\,|\omega_8\rangle + \sin\theta\,|\omega_1\rangle,$$

$$|\phi\rangle = -\sin\theta\,|\omega_8\rangle + \cos\theta\,|\omega_1\rangle,$$

   where $|\omega_8\rangle$ and $|\omega_1\rangle$ are the eight member of the octet and the singlet of *SU*(3) respectively, show that

$$\tan^2\theta = \frac{M_\phi^2 - M_{\omega_8}^2}{M_{\omega_8}^2 - M_\omega^2}.$$

Solutions

1. A symmetric tensor with p indices has $\frac{1}{2}(p+1)(p+2)$ independent components. Thus, a tensor with p upper symmetric indices and q lower symmetric indices has

$$\frac{1}{4}(p+1)(p+2)(q+1)\ (q+2)$$

independent elements. The condition of the tensor being traceless it imposes

$$\frac{1}{4}\,p(p+1)q(q+1)$$

relations among these elements. The dimension $D(p, q)$ for the irreducible tensor with p upper indices and q lower indices, therefore, is

$$\frac{1}{4}(p+1)(p+2)(q+1)(q+2)-\frac{1}{4}\,p(p+1)q(q+1)=\frac{1}{2}(p+1)(q+1)(p+q+2).$$

5

QUARK MODEL OF HADRONS *SU*(6)

Let us now turn from a discussion of symmetries and their applications in particle physics to the spectrum of observed states. To deduce some order in this spectrum, we will introduce a specific model for the hadrons, viz., the quark model.

5.1 *SU*(3) of Flavour, *SU*(2) of Spin and the *SU*(6)

In the 1960s with the help of $SU(3)$ symmetry, it was seen that the observed baryons can be placed in the singlet, octet and decuplet representations of this symmetry, while all the mesons lie in the singlets and the octets. This can easily be explained if we assume, as was done by Gell-Mann and Zweig independently in 1964, that the hadrons are composed of more elementary constituents, called quarks. These quarks come in three flavours u, d, s and belong to the fundamental representation 3 of $SU(3)$. Their antiparticles, the antiquarks $(\bar{u}, \bar{d}, \bar{s})$, are assigned to the conjugate $\bar{R}(\bar{3})$ representation.

Baryons, qqq

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10. \tag{5.1}$$

Mesons, $q\bar{q}$

$$3 \otimes \bar{3} = 1 \oplus 8. \tag{5.2}$$

We don't consider charm and other flavours at present, postponing them to later sections. Baryons have spin 1/2 and 3/2 and mesons 0 and 1. To give appropriate spin to all the particles, it is necessary that each quark has spin 1/2. We then have for spin addition:

(a) qqq

$$\vec{\frac{1}{2}} + \vec{\frac{1}{2}} + \vec{\frac{1}{2}} = \vec{\frac{1}{2}} + \vec{\frac{1}{2}} + \vec{\frac{3}{2}}, \tag{5.3}$$

or

$$2 \otimes 2 \otimes 2 = 2 \oplus 2 \oplus 4. \tag{5.4}$$

(b) $q\bar{q}$

$$\vec{\frac{1}{2}} + \vec{\frac{1}{2}} = \vec{0} + \vec{1}$$

or

$$2 \otimes 2 = 1 \oplus 3. \tag{5.5}$$

We find natural places for baryons (spin 1/2) and baryon resonances (spin 3/2) as well as for the pseudoscalar mesons (spin 0) and vector meson resonances (spin 1). Since quark is assumed to have spin 1/2, our basic constituents can have six different states (spin up and spin down for each flavour). We then look for a group for which this is the fundamental representation, the natural candidate being $SU(6)$. The quark wave function in $SU(6)$ is a product of quark wave function in $SU(3)$ of flavours and $SU(2)$ of spin, i.e.

$$q\{SU(6)\} \sim q\{SU(3)\ u, d, s \text{ flavours}\}q\{SU(2)_S\}.$$

We would like to know what the lowest lying representations of $SU(6)$ are:

$$6 \otimes \bar{6} = 1 \oplus 35,$$

with the $SU(3) \times SU(2)$ content

$$35 = (8, 1) + (8, 3) + (1, 3), \tag{5.6}$$

and we expect the pseudoscalar mesons and vector mesons to belong to 35 of $SU(6)$.

Baryons, qqq

$$6 \otimes 6 \otimes 6 = \underset{A}{20} \oplus \underset{M}{70} \oplus \underset{M}{70} \oplus \underset{S,}{56,} \tag{5.7}$$

with the $SU(3) \otimes SU(2)$ content

$$20 = (8, 2) + (1, 4), \quad 70 = (8, 4) + (8, 2) + (10, 2) + (1, 2), \quad 56 = (10, 4) + (8, 2),$$

where we have labelled the symmetry properties by A (antisymmetric), M (mixed), S (symmetric), and the $SU(3) \otimes SU(2)$ representations by ($SU(3)$ IR, $2S + 1$). The nucleon and Δ fall nicely into the 56, but we note that this has a totally symmetric wave function.

5.2 Quark Statistics and Colour

5.2.1 Problem with the Naive Quark Model

The $SU(6)$ representation 56 to which the baryons belong is symmetric under the interchange of any pair of quark labels, whereas we should have antisymmetric wave function for the three fermion state. We assume, of course, that the ground states have no internal orbital angular momentum.

To solve this problem, we introduce a new degree of freedom, namely colour. We assume that there are three colours, say Red, Green and Blue, which belong to representation 3 of an $SU(3)$ colour. Each quark then comes in three colours and the strong interactions are supposed to be invariant under the transformations of this group. It is further assumed that all the observed hadronic states are colour singlets. The wave function of hadrons is a product:

$$\psi(\text{hadron}) = \psi(SU(6)) \times \psi(SU(3)_{\text{colour}}),$$

$$(3 \otimes 3 \otimes 3)_{\text{colour}} = (1 \oplus 8 \oplus 8 \oplus 10)_{\text{colour}},$$

and we know that 1 is totally antisymmetric and so (hadron) has overall antisymmetry. Let us investigate the evidence for the colour quantum number.

5.2.2 $e^+ e^- \to$ Hadrons

This reaction is believed to proceed via the electromagnetic production of a $q\bar{q}$ pair which then interacts to produce hadrons. By comparing this reaction with production, we define a quantity R which is given by

$$R = \frac{\sigma(e^+ e^- \to \text{hadrons})}{\sigma(e^+ e^- \to \mu^+ \mu^-)} = \frac{\sum_i Q_i^2}{1} = \sum_i Q_i^2 , \tag{5.8}$$

i runs over all the quark varieties. Without colour we would expect for three quarks,

$$R = (2/3)^2 + (-1/3)^2 + (-1/3)^2 = 2/3.$$

However, in the region of 2 ~ 3 GeV in the centre of mass, the observed value of R is ~ 2 consistent with quarks coming in three colours. Above 3 ~ 4 GeV the situation is complicated by the excitation of further flavours, charm, etc. At energies > 10 GeV centre of mass, the measurements from the PETRA collider at DESY, HAMBURG, FRG have the constant value of 11/3. At this energy the u, d, s, c, and b quarks can contribute and the square of charges add to 11/9. If we have three colours, this number is to be multiplied by three to give 11/3. The excellent agreement gives direct evidence for colour.

5.2.3 $\pi^0 \to 2\gamma$

This decay was calculated by J. Steinberger in 1949 in the proton-antiproton approximation represented by Fig. 5.1. The matrix element is obtained in terms of the coupling constant g of the pseudoscalar pion-nucleon coupling and the proton charge e (taking proton as elementary). The result for the decay rate was

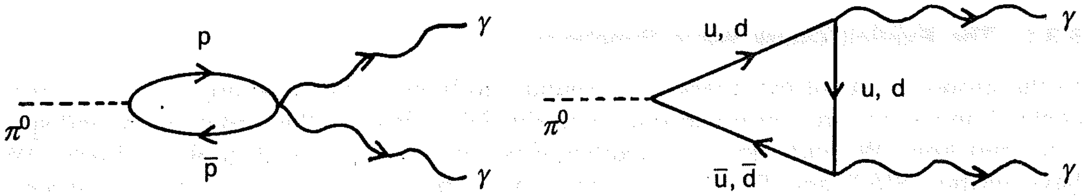


Fig. 5.1 $\pi^0 \to 2\gamma$.

$$\Gamma = \frac{\alpha^2}{16\pi^2} \frac{g^2}{4\pi^2} \frac{m_\pi^3}{m_p^2} = 13.8 \text{ eV} , \tag{5.9}$$

to be compared with the experimental value of 7.8 + 1.0 eV. Now that we know that the fundamental constituents of π^0 are quarks, we need to redo the calculations replacing protons in the internal lines of the diagram by quarks. The π^0 contains only u and d quarks, so we require the coupling of pions to the doublet of quarks $q = \begin{pmatrix} u \\ d \end{pmatrix}$ in the form $g_q \bar{q} \gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi} q$ identical to the coupling of pions and nucleons. For the π^0, we have $g_q \bar{q} \gamma_5 \tau_3 q \pi^0$. The proton's constituents are *uud*, so the coupling of the π^0 to the proton is equal to $g_q(1 + 1 - 1) = g_q$, the numbers in the bracket being the values of $\tau_3 = 2I_3$ for the quarks (+1 for *u* and −1 for *d*). Thus, we have $g_q = g$, the usual pion-nucleon coupling constant. The use of quarks in the diagram has the effect of replacing αg by $\alpha g \sum_{i=u,d} (2I_3)_i Q_i^2$ $= \alpha g\{(2/3)^2 - (1/3)^2\} = \alpha g/3$. For the rate, therefore, we have the replacement $\alpha^2 g^2 \to (1/9)\alpha^2 g^2$, and so the rate is too small by a factor of 5–6. If, however, quarks come in three colours and π^0 being a colour singlet, couples equally strongly to each other αg in the matrix element is replaced by $3 \times \{(1/3)\alpha g\} = \alpha g$, yielding the same result as obtained by J. Steinberger (1949). A more sophisticated calculation involving current algebra by S.L. Adler gave a rate $\Gamma(\pi \to 2\gamma) = 7.29$ eV when colour is included. This is in excellent agreement with experiment. Again we require three colours to get good agreement.

5.3 Baryon

Let us discuss baryons from the point of view of *SU*(6) and the quark model. The baryons are three quark states and so we expect them to belong to the irreducible representations 20, 70, and 56. The 56 representation is completely symmetric and contains the *SU*(3) octet of $1/2^+$ baryons and decuplet of $3/2^+$ baryon resonances.

We assume that all baryons will be described by symmetric quark wave function, the necessary antisymmetry being provided by the colour part of the wave function (all baryons being colour singlets).

5.3.1 The Explicit Quark Wave Functions

In the ground state all the quarks are assumed to have zero orbital angular momentum (*s*-state) and so we are concerned only with the *SU*(3) flavour (the unitary spin) and spin wave functions. We write down the explicit wave functions for the decuplet and octet. We have written *SU*(2) and *SU*(3) wave functions of baryon in terms of their constituent quarks in Section 4.7. To give a complete description of a baryon, it is necessary to specify the spin wave functions as well. The spin 3/2 wave functions are:

$$\left.\begin{aligned}
|s = 3/2, s_z = 3/2\rangle &= \uparrow\uparrow\uparrow \\
|s = 3/2, s_z = 1/2\rangle &= \frac{1}{\sqrt{3}}\,[\uparrow\uparrow\downarrow + (\uparrow\downarrow + \downarrow\uparrow)\uparrow], \\
|s = 3/2, s_z = -1/2\rangle &= \frac{1}{\sqrt{3}}\,[(\uparrow\downarrow + \downarrow\uparrow)\downarrow + \downarrow\downarrow\uparrow], \\
|s = 3/2, s_z = -3/2\rangle &= \downarrow\downarrow\downarrow,
\end{aligned}\right\} \qquad (5.10)$$

while for spin 1/2, they are:

$$\left.\begin{aligned}
\chi^{+}_{M,A} &= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow,\\
\chi^{-}_{M,A} &= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow,\\
\chi^{+}_{M,S} &= \frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow],\\
\chi^{-}_{M,S} &= \frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\downarrow - 2\downarrow\downarrow\uparrow].
\end{aligned}\right\} \tag{5.11}$$

To write the hadron wave functions in terms of the quark wave functions with their spins, we consider the simpler case of $SU(4)$ of isospin and spin.

5.4 *SU*(4) of Isospin and Spin

Let us consider a doublet u, d of isospin. This doublet is the fundamental representation of the isospin $SU(2)$ group. Since u and d are spin 1/2 objects, they both have two states of spin, so that we are dealing with four states, $u\uparrow$, $u\downarrow$ and $d\uparrow$, $d\downarrow$. These four states may be regarded as the fundamental representation of a group of transformations, $SU(4)$. If we form states from quark and antiquark, we shall get states with spin 0 and those with spin 1. Obviously, these spin 0 and spin 1 states will belong to the same irreducible representation of $SU(4)$ of isospin and spin, and hence their properties will be related. Similarly, if we consider states with three quarks, we shall get states with spin 1/2 and those with spin 3/2 and their properties will also be related. An important consequence of this isospin-spin symmetry is that the ratio of the magnetic moment of proton to that of neutron is $-3/2$, whereas the experimental value is -1.46 in excellent agreement with the prediction. In terms of quarks and antiquarks, the normalized isospin eigenstates belonging to the triplet representation of $SU(2)_I$ are given in (4.26).

Now u and d are spin 1/2 objects and hence belong to the fundamental doublet representation of the group $SU(2)_S$, where the subscript stands for spin. If we now take into account the spin of the u, d also and keep in mind that the pions are spin 0 particles, the wave function of these mesons in terms of its constituents along with their spins can be written as,

$$I = 1 \quad \begin{cases}
u\bar{d}\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow), & I_z = 1,\\[2ex]
\dfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow), & I_z = 0,\\[2ex]
d\bar{u}\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow), & I_z = -1.
\end{cases}$$

This is the irreducible representation (3, 1) of the group $SU(2)_I \otimes SU(2)_S$ which is a subgroup of $SU(4)$.

The ρ mesons are spin 1 states, and hence their wave functions in terms of the constituents are: For $S = 1$, $S_z = 1$,

$$I = 1 \begin{cases} u\bar{d}\ \uparrow\uparrow, & I_z = 1, \\ \dfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\uparrow\uparrow, & I_z = 0, \\ d\bar{u}\uparrow\uparrow, & I_z = -1. \end{cases}$$

For $S = 1$, $S_z = 0$,

$$I = 1 \begin{cases} u\bar{d}\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), & I_z = 1, \\ \dfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), & I_z = 0, \\ d\bar{u}\,\dfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), & I_z = -1. \end{cases}$$

For $S = 1$, $S_z = -1$,

$$I = 1 \begin{cases} u\bar{d}\downarrow\downarrow, & I_z = 1, \\ \dfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})\downarrow\downarrow, & I_z = 0, \\ d\bar{u}\downarrow\downarrow & I_z = -1. \end{cases}$$

Similarly, we can write the wave function for the isospin singlet of pseudoscalar and vector mesons with one- and three-states respectively. We have thus (3, 3), (3, 1), (1, 3) and (1, 1) representations of the group $SU(2)_I \otimes SU(2)_S$. The first three belong to the representation 15 of the group $SU(4)$ of isospin and spin, and the last one to the singlet.

For the four Δ resonances belonging $I = 3/2$, the wave functions are symmetric under the interchange of any pair of quarks and given in (4.30). Again, if we take into account the spin of the constituents, we obtain for the wave functions, for $S = 3/2$, $S_z = 3/2$, the relations

$$|3/2, 3/2\rangle = \Delta^{++} = uuu\ \uparrow\uparrow\uparrow,$$

$$|3/2, 1/2\rangle = \Delta^{+} = \frac{1}{\sqrt{3}}(uud + udu + duu)\ \uparrow\uparrow\uparrow,$$

$$|3/2, -1/2\rangle = \Delta^{0} = \frac{1}{\sqrt{3}}(udd + dud + ddu)\ \uparrow\uparrow\uparrow,$$

$$|3/2, -3/2\rangle = \Delta^{-} = ddd\ \uparrow\uparrow\uparrow.$$

Similarly, we can write the wave function for the remaining three spin states. These sixteen states form the irreducible representation (4, 4) of $SU(2)_I \otimes SU(2)_S$.

For proton and neutron belonging to the doublet representation of the $SU(2)_I$, the wave functions in terms of the constituents are given in (4.36), (4.38) and (4.39). If we now include the spin of the constituents, we can write for the spin up states

$$p = \frac{1}{\sqrt{2}}(ud - du)u \ \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow,$$

$$n = \frac{1}{\sqrt{2}}(ud - du)d \ \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow.$$

And

$$|p\rangle = \frac{1}{\sqrt{6}}(2uud - udu - duu) \ \frac{1}{\sqrt{6}}(2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow),$$

$$|n\rangle = -\frac{1}{\sqrt{6}}(2ddu - dud - udd) \ \frac{1}{\sqrt{6}}(2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow).$$

Similarly, we can write the wave functions for the spin down states. We note that there are two ways of writing down the wave function for proton and neutron in terms of the constituent states. We add the two wave functions with equal weight of 1/2 each to get the wave function for the proton as given in (5.15).

These (4, 4) and (2, 2) states of the delta resonances and nucleons correspond to the completely symmetric 20 representation of the group $SU(4)$ of isospin and spin. Therefore, $SU(4)$ relates the properties of delta resonances and of nucleon. For example, the magnetic transition moment of delta resonance to nucleon gets related to the proton magnetic moment.

Also, the magnetic moments of proton and neutron get related through

$$\frac{\mu_p}{\mu_n} = -1.5.$$

This is to be compared with the experimental value of 2.79/– 1.92 = – 1.46 which is in excellent agreement with the prediction.

5.5 Decuplet (3/2⁺ Baryon Resonances)

We can, therefore, write the wave functions for Δ in definite spin states as

$$\left.\begin{aligned}
\Delta^{++}(J_z = 3/2) &= u_1u_2u_3 \ 1\uparrow 2\uparrow 3\uparrow, \\
\Delta^{++}(J_z = 1/2) &= u_1u_2u_3 \ \frac{1}{\sqrt{3}}[\uparrow\uparrow\downarrow + (\uparrow\downarrow + \downarrow\uparrow)\uparrow], \\
\Delta^{++}(J_z = -1/2) &= u_1u_2u_3 \ \frac{1}{\sqrt{3}}[(\uparrow\downarrow + \downarrow\uparrow)\downarrow + \downarrow\downarrow\uparrow], \\
\Delta^{++}(J_z = -3/2) &= u_1u_2u_3 \ 1\downarrow 2\downarrow 3\downarrow, \ldots
\end{aligned}\right\} \qquad (5.12)$$

and so on. The wave functions are symmetric in spin and flavour (isospin). The quarks are supposed to be in s-state so that the wave function (completely symmetric in spin and flavour) is in conflict with the generalized Pauli principle. Since quarks carry colour quantum number, the total wave function can be made antisymmetric by multiplying it with totally antisymmetric colour wave function. In general, for $J^P = 3/2^+$,

$$\left.\begin{aligned} D(p,p,p) &= p\uparrow p\uparrow p\uparrow, \\ D(p,q,q) &= \frac{1}{\sqrt{3}}(p\uparrow q\uparrow q\uparrow + q\uparrow p\uparrow q\uparrow + q\uparrow q\uparrow p\uparrow), \\ D(p,q,r) &= \frac{1}{\sqrt{6}}(p\uparrow q\uparrow r\uparrow + q\uparrow r\uparrow p\uparrow + r\uparrow p\uparrow q\uparrow + q\uparrow p\uparrow r\uparrow \\ &\quad + p\uparrow r\uparrow q\uparrow + r\uparrow q\uparrow p\uparrow) \end{aligned}\right\} \tag{5.13}$$

where p, q, r denote quark flavours.

5.6 Octet of Baryons

We can construct wave function for the nucleon, say proton with $J_z = 1/2$, in several ways. The simplest, perhaps, is to start with u and d quarks having $S = S_z = 0$. This is antisymmetric in spin so that we antisymmetrize in flavour also to obtain overall symmetry of the two quarks. An explicit wave function which corresponds to p, the $I = 1/2$, $I_z = 1/2$ and $Y = 1$ state of an octet is

$$\frac{1}{\sqrt{2}}(u_1d_2 - d_1u_2)u_3.$$

This is antisymmetric in flavour of the first two quarks. Because

$$\text{spin } 0 = \frac{1}{\sqrt{2}}(1\uparrow 2\downarrow - 1\downarrow 2\uparrow),$$

the two quark symmetric wave function is

$$\frac{1}{2}(u\uparrow d\downarrow - d\uparrow u\downarrow - u\downarrow d\uparrow + d\downarrow u\uparrow). \tag{5.14}$$

If we now add a third quark, $u\uparrow$, a reasonable wave function for the proton with spin up is

$$\Psi_{[1,2]3} = \frac{1}{2}(u_1d_2 - d_1u_2)\, u_3\, (1\uparrow 2\downarrow - 1\downarrow 2\uparrow)3\uparrow, \tag{5.14a}$$

but this is not completely symmetric. A final satisfactory wave function can be obtained by symmetrizing

$$\Psi(p, J_z = \frac{1}{2}) = A[\Psi_{[1,2]3} + 1 \leftrightarrow 3 + 2 \leftrightarrow 3],$$

i.e.

$$A[\Psi_{[1,2]3} + \Psi_{[3,2]1} + \Psi_{[1,3]2}].$$

The normalized wave function of proton with spin up can then be written as

$$\begin{aligned}|p\uparrow\rangle = \frac{1}{\sqrt{18}}[&2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow\\ &- d\uparrow u\downarrow u\uparrow + 2d\downarrow u\uparrow u\uparrow - d\uparrow u\uparrow u\downarrow\\ &- u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + 2u\uparrow u\uparrow d\downarrow].\end{aligned} \tag{5.15}$$

For the neutron with spin up, we have

$$\begin{aligned}|n\uparrow\rangle = -\frac{1}{\sqrt{18}}[&2d\uparrow u\downarrow d\uparrow - d\uparrow u\uparrow d\downarrow - d\downarrow u\uparrow d\uparrow\\ &- u\uparrow d\downarrow d\uparrow + 2u\downarrow d\uparrow d\uparrow - u\uparrow d\uparrow d\downarrow\\ &- d\uparrow d\downarrow u\uparrow - d\downarrow d\uparrow u\uparrow + 2d\uparrow d\uparrow u\downarrow.\end{aligned} \tag{5.15a}$$

Equivalently, if we start with

$$|p\uparrow\rangle = \frac{1}{\sqrt{2}}(\phi^P_{M,A}\chi^+_{M,A} + \phi^P_{M,S}\chi^+_{M,S}), \tag{5.16}$$

we get the same proton wave function. In general,

$$\begin{aligned}B(p,q,q) = \frac{1}{\sqrt{18}}[&p\uparrow q\downarrow q\uparrow + q\downarrow q\uparrow p\uparrow + q\downarrow p\uparrow q\uparrow\\ &+ p\uparrow q\uparrow q\downarrow + q\uparrow p\uparrow q\downarrow + q\uparrow q\downarrow p\uparrow)\\ &- 2(p\downarrow q\uparrow q\uparrow + q\uparrow p\downarrow q\uparrow + q\uparrow q\uparrow p\downarrow)].\end{aligned} \tag{5.17}$$

$$\begin{aligned}B(p,q,r) = \frac{1}{6}[&p\uparrow q\downarrow r\uparrow + q\downarrow r\uparrow p\uparrow + r\uparrow p\uparrow q\downarrow + q\downarrow p\uparrow r\uparrow + p\uparrow r\uparrow q\downarrow\\ &+ r\uparrow q\downarrow p\uparrow + p\downarrow q\uparrow r\uparrow + q\uparrow r\uparrow p\downarrow + r\uparrow p\downarrow q\downarrow + q\uparrow p\downarrow r\uparrow\\ &+ p\downarrow r\uparrow q\uparrow + r\uparrow q\uparrow p\downarrow - 2(r\downarrow p\uparrow q\uparrow + q\uparrow r\downarrow p\uparrow\\ &+ p\uparrow q\uparrow r\downarrow + r\downarrow q\uparrow p\uparrow + q\uparrow p\uparrow r\downarrow + p\uparrow r\downarrow q\uparrow)].\end{aligned} \tag{5.18}$$

$$\begin{aligned}B'(p,q,r) = \frac{1}{\sqrt{12}}[&p\uparrow q\downarrow r\uparrow + q\downarrow r\uparrow p\uparrow + r\uparrow p\uparrow q\downarrow + q\downarrow p\uparrow r\uparrow\\ &+ p\uparrow r\uparrow q\downarrow + r\uparrow q\downarrow p\uparrow - (p\downarrow q\uparrow r\uparrow + q\uparrow r\uparrow p\downarrow\\ &+ r\uparrow p\downarrow q\uparrow + q\uparrow p\downarrow r\uparrow + p\downarrow r\uparrow q\uparrow + r\uparrow q\uparrow p\downarrow)].\end{aligned} \tag{5.19}$$

5.7 Mesons

The fundamental triplet (u, d, s) is multiplied by antitriplet $(\bar{u}, \bar{d}, \bar{s})$, giving us nine different combinations

$$(u\bar{u}, u\bar{d}, u\bar{s}, d\bar{u}, d\bar{d}, d\bar{s}, s\bar{u}, s\bar{d}, s\bar{s}).$$

While the six charge states are unique, the splitting of the three neutral combinations corresponding to $S = 0$ may more easily be seen by considering the subgroup $SU(2)$. The isodoublet transforms in the same manner as does its conjugate. This was seen by considering the transition described by the G parity: $G = Ce^{i\pi I_2}$, the product $3 \otimes \bar{3}$, then gives three antisymmetric states with respect to G and the one symmetric state.

$$\left.\begin{array}{l} \text{Antisymmetric } \left(|u\bar{d}\rangle, \quad |\bar{u}d\rangle, \quad \frac{1}{\sqrt{2}}|(u\bar{u} - d\bar{d})\rangle \right); \\[2ex] \text{Symmetric } \frac{1}{\sqrt{2}}|(u\bar{u} + d\bar{d})\rangle . \end{array}\right\} \tag{5.20}$$

If we identify the isotriplet with the π-meson and now introduce the third (strange) quark, the new state could be constructed only from the G-symmetric isosinglet state. The symmetric $q\bar{q}$ combination is $(1/\sqrt{3})|u\bar{u} + d\bar{d} + s\bar{s}\rangle$. $1/\sqrt{3}$ is the normalization factor. It is identified with the heavier isosinglet, the $\eta'(958)$. The orthonormality can be used to fix the $SU(3)$ wave function of the third neutral (second isosinglet). Let

$$|\eta\rangle = \frac{1}{N}(|uu\rangle + |dd\rangle + a|ss\rangle).$$

Then the requirement $<\eta|\eta\rangle = 1$ and $<\eta|\eta'\rangle = 0$ gives $a = -2$ and $N = 6$.

5.7.1 Ground States of Ordinary Mesons

The pseudoscalar mesons

The ground state pseudoscalar mesons have

$$L = 0, \qquad S = 0, \qquad J = 0, \qquad P = -1.$$

Hence they are in the ${}^1S_0$ state. Their quark content and some properties are given in Table 5.1.

Table 5.1 Some Properties and Quark Content of Pseudoscalar Mesons

Particle	*Mass in* MeV	*I*	I_z	*Y*	*Quark content*
π^+	140	1	+1	0	$u\bar{d}$
π^0	135	1	0	0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
π^-	140	1	−1	0	$d\bar{u}$
K^+	494	1/2	+1/2	1	$u\bar{s}$
K^0	494	1/2	−1/2	1	$d\bar{s}$
$\bar{K}^0$	494	1/2	+1/2	−1	$s\bar{d}$
K^-	494	1/2	−1/2	−1	$s\bar{u}$
η	549	0	0	0	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$

The vector mesons

For the vector mesons

$$L = 0, \quad S = 1, \quad J = 1, \quad P = -1.$$

Hence they are in the $^3S_0$ state. Their quark content and some properties are given in Table 5.2.

Table 5.2 Some Properties and Quark Content of Vector Mesons

Particle	*Mass in* MeV	I	I_z	Y	*Quark content*
ρ^+	770	1	+1	0	$u\bar{d}$
ρ^0	770	1	0	0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
ρ^-	770	1	–1	0	$d\bar{u}$
K^{*+}	892	1/2	+1/2	1	$u\bar{s}$
K^{*0}	892	1/2	–1/2	1	$d\bar{s}$
$\bar{K}^{*0}$	892	1/2	+1/2	- 1	$s\bar{d}$
K^{*-}	892	1/2	–1/2	- 1	$s\bar{u}$
ω	783	0	0	0	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$
ϕ	1019	0	0	0	$s\bar{s}$

5.8 Applications of Quark Model

As a first use of these wave functions, we consider the static electromagnetic properties of baryons. We picture these as arising from those of the constituent quarks. Since the hadrons are bound in relative s state, the magnetic moment μ of the hadrons, for example, is the sum of the constituent quark magnetic moments. We assume that the individual quark moments are proportional to the charges q_i in the units of e, the electronic charge, then

$$\mu\,(\text{hadron}) = \sum_i \boldsymbol{\mu}_i\,(\text{quarks}). \tag{5.21}$$

The magnetic moment for the particle of spin J is

$$\begin{aligned} \mu &= \langle J_z = J \,|\, \mu_z \,|\, J_z = J\rangle \\ &= \langle J_z = J \,|\, \sum_i \frac{Q_i e}{2m_i}\,\sigma_z^{(i)} \,|\, J_z = J\rangle, \end{aligned} \tag{5.22}$$

where we have taken

$$\boldsymbol{\mu}_i = \frac{Q_i e}{2m_i}\,\boldsymbol{\sigma}^{(i)}. \tag{5.23}$$

Setting $m_u = m_d$, we calculate the values of the baryon magnetic moments with the aid of the explicit wave functions (see Problem 3 at the end of the chapter).

5.8.1 Solution for Proton

$$\mu = \sum_i \frac{Q_i e}{2m_i}\,\sigma_3^i = \frac{e}{2}\left(\frac{2}{3}\frac{\sigma_{u,3}}{m_u} - \frac{1}{3}\frac{\sigma_{d,3}}{m_d} - \frac{1}{3}\frac{\sigma_{s,3}}{m_s}\right)$$

$$= \frac{e}{2m}\left(\frac{2}{3}\sigma_{u,3} - \frac{1}{3}\sigma_{d,3} - \frac{1}{3}y\sigma_{s,3}\right), \tag{5.24}$$

where

$$y = m/m_S; \qquad m = \frac{1}{2}(m_u + m_d). \tag{5.25}$$

We may rewrite the proton wave function

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}[\,|uud\rangle\,(2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) + \text{cyclic permutations}], \tag{5.26}$$

$$\langle p\uparrow|\sum_i Q_i\sigma_3^{(i)}|p\uparrow\rangle = \frac{1}{18}\left[4\left\{\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right\}+\left\{\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right\}+\left\{-\frac{2}{3}+\frac{2}{3}-\frac{1}{3}\right\}\right]\times 3$$

$$= \frac{1}{6}\left[\frac{20}{3}-\frac{1}{3}-\frac{1}{3}\right] = 1, \tag{5.27}$$

$$\langle p\uparrow|\frac{e}{2mc}\sum_i Q_i\sigma_3^{(i)}|p\uparrow\rangle = \frac{e}{2m}. \tag{5.28}$$

Similarly,

$$\langle n\uparrow|\frac{e}{2mc}\sum_i Q_i\sigma_3^{(i)}|n\uparrow\rangle = -\frac{2}{3}\frac{e}{2m}, \tag{5.29}$$

$$\mu_p/\mu_n = -\frac{3}{2}, \tag{5.30}$$

which is to be compared with the experimental ratio

$$\mu_p/\mu_n = -\frac{2.793}{1.913} = -1.46. \tag{5.31}$$

The wave function for Λ

$$|\Lambda, 1/2\rangle = \frac{1}{2\sqrt{3}}\,[(|uds\rangle - |dus\rangle)\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + (|sud\rangle - |sdu\rangle)\,(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow)$$
$$+ (|dsu\rangle - |usd\rangle)\,(\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow)] \tag{5.32}$$

gives

$$\mu_\Lambda = -\frac{1}{3}y\frac{e}{2m}, \tag{5.33}$$

where

$$y = m/m_S. \tag{5.34}$$

The comparison of these theoretical values with experiment is shown in Table 5.3.

Table 5.3 Magnetic Moments of Baryon Octet

Baryon	μ_B	*Predicted value* (n·m)	*Observed value* (n·m)
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	2.79	2.793
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	− 1.86	− 1.913
Λ	μ_s	− 0.58	− 0.613 ± 0.004
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.69	2.458 ± 0.010
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	− 1.05	− 1.16 ± 0.025
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	− 1.40	− 1.25 ± 0.014
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	− 0.47	− 0.6507 ± 0.0025

5.9 Hadron Masses

We now discuss the subject of hadron masses within a picture of hadrons as being composed of quarks. More particularly, we will work in a constituent model where quarks are bound by an effective potential due to the exchange of coloured gluons. This QCD inspired model has provided many insights into the pattern of hadron masses which are assumed to come from the quark masses and the spin-spin interaction. According to De Rujula et al., all spin effects arise from the short distance one gluon exchange interaction. The spin-spin interaction contribution is responsible for $\Delta - N$, $\Sigma^* - \Sigma$, $\Sigma - \Lambda$, etc. mass differences for baryons and $\rho - \pi$, $K^* - K$, etc. mass splittings among mesons. In the non-relativistic situation, the spin-spin interaction contribution to the mass has the form

$$\Delta M_{\text{s.s}} = c \sum_{i<j} \frac{\mathbf{s}_i \cdot \mathbf{s}_j}{m_i m_j} \tag{5.35}$$

Σ-hyperon: u, d are in the symmetric $I = 1$ state and so in the $S = 1$ symmetric state. The total spin of all the three quarks is 1/2. We have

$$\mathbf{s}_u \cdot \mathbf{s}_d = 1/4, \qquad \mathbf{s}_u \cdot \mathbf{s}_s + \mathbf{s}_d \cdot \mathbf{s}_s = -1. \tag{5.36}$$

Exercise Prove (5.36).

For the Σ-hyperon, we have

$$M_\Sigma = m_u + m_d + m_s + c\left[\frac{\mathbf{s}_u \cdot \mathbf{s}_d}{m_u^2} + \frac{(\mathbf{s}_u + \mathbf{s}_d) \cdot \mathbf{s}_s}{m_u m_s}\right]$$

$$= m_u + m_d + m_s + c\left[\frac{1}{4m_u^2} - \frac{1}{m_u m_s}\right] \tag{5.37}$$

For the Λ hyperon, u, d are in the antisymmetric $I = 0$ and antisymmetric $S = 0$ state. Again the total spin of all the three quarks is 1/2. We now have

$$\mathbf{s}_u \cdot \mathbf{s}_d = -3/4, \qquad (\mathbf{s}_d + \mathbf{s}_u) \cdot \mathbf{s}_s = 0 \tag{5.38}$$

giving

$$\begin{aligned} M_\Lambda &= m_u + m_d + m_s + c\left[\frac{\mathbf{s}_u \cdot \mathbf{s}_d}{m_u^2} + \frac{(\mathbf{s}_u + \mathbf{s}_d)\cdot \mathbf{s}_s}{m_u m_s}\right] \\ &= m_u + m_d + m_s - \frac{3c}{4m_u^2}, \end{aligned} \tag{5.39}$$

So

$$M_\Sigma - M_\Lambda = \frac{c}{m_u^2}\left(1 - \frac{m_u}{m_s}\right). \tag{5.40}$$

Since strange quark is heavier than the up (or down) quark, we have $M_\Sigma > M_\Lambda$ in agreement with experiment.

We may note here that the masses of Σ_c and Λ_c would differ by the same mechanism. With the charm quark replacing the strange quark,

$$M_{\Sigma_c} - M_{\Lambda_c} = \frac{c}{m_u^2}\left(1 - \frac{m_u}{m_c}\right) \tag{5.41}$$

Now, since the charm quark is heavier than the strange one, this mass difference would be even larger than between the Σ and Λ. Again this prediction seems correct as we will see in more detail in Chapter 10.

PROBLEMS

1. Work out the explicit $SU(6)$ wave functions for all the remaining members of the baryon octet. The expression for proton is given.
2. Use expressions in Problem 1 to obtain the magnetic moments of octet baryons in terms of those of quarks.
3. In the $SU(3)$ symmetry limit, the quark magnetic moments are proportional to their charges:

$$\mu_d = \mu_S = -\frac{1}{2}\mu_u$$

This amounts to assuming that u, d, and s quarks all have the same mass. The value of the proton magnetic moment μ_P is 2.793 n · m. Calculate the values of the magnetic moments of the other members of the baryon octet. Compare with the measured values given in the particle data group. How would the result change if we assign $m_u = m_d = 300$ MeV and $m_s = 500$ MeV?

Furthermore, values of m and m_s determined from magnetic moments match reasonably well with those determined from masses.

Now, assume that the magnetic moment of a quark is given by

$$\mu_i = \frac{e_i}{2m_i c},$$

where i stands for the flavours u, d, s, of the quark. Compute the magnetic moments of the octet baryons. Take $m_u = m_d$.

4. Compute the magnetic moments of Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-} and Ω^{-}. Show that the transition magnetic moment is

$$\mu_{\Delta + p} = \langle \Delta^{+} \, | \mu_z | \, p \rangle = \frac{2\sqrt{2}}{3} \mu_p.$$

5. Calculate the ratio

$$\Gamma(D^{*0} \to D^0\gamma)/\Gamma(D^{*+} \to D^{+}\gamma)$$

with the approximation $\mu_C = 0$.

Recalculate, taking $\mu_C = -\frac{2}{3}\mu_S = .41$. Assume the masses of the charmed-strange mesons D_S and D_S^* as 2030 MeV/c^2 and 2140 MeV/c^2 respectively. Using the values of μ_C as given above, estimate the width for the decay $D_S^* \to D_S\gamma$.

6. Discuss the significance of

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

Obtain its theoretical expression and discuss its agreement with experiment.

7. The static quark model is also useful in obtaining relations among cross-sections. By treating the following processes as due to the scattering of quarks, derive the following relations between cross-sections at a given energy:

(a) $\sigma(\Lambda p) = \sigma(pp) + \sigma(K^- n) - \sigma(\pi^+ p)$,

(b) $\sigma(\Sigma^- p) = \sigma(pp) + \sigma(K^- p) - \sigma(\pi^- p) + 2[\sigma(K^+ n) - \sigma(K^+ p)]$,

(c) $\sigma(\Sigma^- n) = \sigma(pp) + \sigma(K^- p) - \sigma(\pi^- p)$.

6

HADRON CURRENTS

6.1 Introduction

Hadron currents are of great importance as they appear explicitly in the electromagnetic and in the weak interactions. Of all the currents one deals with in particle physics, the electromagnetic is no doubt the most familiar. The coupling of hadrons to the electromagnetic field operator A_μ is described, to the lowest order, by an interaction Hamiltonian density

$$H^{\mathrm{EM}} = e\, j_\mu^{\mathrm{EM}} A^\mu, \tag{6.1}$$

where j_μ^{EM} is the hadron electromagnetic current. It is a vector operator and depends upon the nature of hadron. Reflecting conservation of electric charge, we assert that the charge operator

$$Q^{\mathrm{EM}} = \int d^3x j_0^{\mathrm{EM}}(x_0, \mathrm{x}) \tag{6.2}$$

is independent of time x_0 and the current j_μ^{EM} is conserved:

$$\partial^\mu j_\mu^{\mathrm{EM}} = 0. \tag{6.3}$$

j_μ^{EM} is taken to be odd under the charge conjugation symmetry transformation. Electromagnetic interactions, of course, conserve baryon number (B) and strangeness (S) or, equivalently, hypercharge $Y = B + S$; and they conserve the third component I_3 of isotopic spin. According to the Gell-Mann-Nishijima formula,

$$Q^{\mathrm{EM}} = I_3 + \frac{Y}{2}. \tag{6.4}$$

The electromagnetic current j_μ^{EM} consists of a part which transforms like a scalar under isospin rotations and a part which transforms like the third component of an isovector, i.e.

$$j_\mu^{\mathrm{EM}} = j_\mu^{\mathrm{EM}}(I = 0) + j_\mu^{\mathrm{EM}}(I = 1). \tag{6.5}$$

If we regard proton and neutron as fundamental particles, then we can take

$$j_\mu^{\mathrm{EM}} = \bar{p}\gamma_\mu p = \frac{1}{2}[\bar{p}\gamma_\mu p - \bar{n}\gamma_\mu n] + \frac{1}{2}[\bar{p}\gamma_\mu p + \bar{n}\gamma_\mu n], \tag{6.6}$$

where p and n stand for the second quantized Dirac field for the proton and neutron, respectively. The electromagnetic current density j_μ^{EM} is written in the unit of electron electric charge. The first term in the current density transforms as the third component of the isospin whereas the second component is a scalar, that is to say that it transforms like isospin zero. In terms of the quarks,

$$j_\mu^{\mathrm{EM}} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s, \tag{6.7}$$

with

$$j_\mu^{\mathrm{EM}}(I=0) = \frac{1}{6}\,[\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\,\bar{s}\gamma_\mu s], \tag{6.8}$$

$$j_\mu^{\mathrm{EM}}(I=1) = \frac{1}{2}\,[\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d]. \tag{6.9}$$

Here again the particle symbols u, d, and s stand for the second quantized Dirac fields for the quarks u, d, and s respectively. The two pieces are separately conserved, implying conservation of hypercharge and third component of isospin, the operators corresponding to which are given respectively by

$$\left.\begin{aligned} \frac{Y}{2} &= \int d^3x j_0^{\mathrm{EM}}(I=0), \\ I_3 &= \int d^3x j_0^{\mathrm{EM}}(I=1). \end{aligned}\right\} \tag{6.10}$$

In terms of the fundamental proton and neutron field operators, the operators Y and I_3 are:

$$\left.\begin{aligned} Y &= \int d^3x[p^\dagger p + n^\dagger n] = \int d^3x[\bar{p}\gamma^0 p + \bar{n}\gamma^0 n], \\ I_3 &= \frac{1}{2}\int d^3x[p^\dagger p - n^\dagger n] = \frac{1}{2}\int d^3x[\bar{p}\gamma^0 p - \bar{n}\gamma^0 n], \end{aligned}\right\} \tag{6.11}$$

whereas in terms of the fundamental quark fields these are

$$\left.\begin{aligned} Y &= \frac{1}{3}\int d^3x[u^\dagger u + d^\dagger d - 2s^\dagger s], \\ I_3 &= \frac{1}{2}\int d^3x\,[u^\dagger u - d^\dagger d]. \end{aligned}\right\} \tag{6.12}$$

6.2 Structure of Hadrons

In the early part of this century, Rutherford studied the structure of atom by scattering alpha particles from thin gold foils. The fact that some alpha particles were scattered through unexpectedly large angles implied that an atom must have at its centre a very massive, point-like positively charged object. That was the discovery of the atomic nucleus.

Since then the scattering experiments have been very usefully employed in studying the structure of hadrons. In particular, we have learnt a great deal from lepton hadron scattering. The measurements of charge distribution in nuclei (via elastic scattering) revealed details of the nuclear structure. The observation of quasi-elastic peaks in electron nucleus scattering verified the presence of nucleons in nuclei.

In the late sixties, history repeated itself when a SLAC-MIT team of scientists performed inelastic electron-proton scattering with incident energies ranging between 7 and 17 GeV at the Stanford Linear Accelerator. To their surprise, the experimenters observed hundreds of times more counts of the outgoing electrons at 6° and 10° angles than expected. They also observed scaling in the scattering. That indicated that in high energy inelastic scattering, the incident electrons sometimes hit hard point-like constituents (partons as they were called by Feynman) inside the proton, just as the incident alpha particles were occasionally scattered by the nucleus in the Rutherford experiment. Tests on quantum numbers of these constituents confirm that they are identical to quarks. The energy conservation sum rules yielded evidence that there are also neutral constituents (the gluons) in the nucleon.

The features of strange and charm particle production gave evidence for the existence of a sea of quark-antiquark pairs in the nucleon. Finally, none of these experiments have provided evidence for free quarks. This has led to the belief that quarks are permanently confined inside hadrons, or at least they are extremely hard to liberate.

Let us now review essential aspects of theory of scattering which we shall be using to study the hadronic structure.

6.2.1 Rutherford Scattering

The cross-section for the elastic scattering of two charged spinless, point particles due to electromagnetic interaction is given by the familiar Rutherford scattering formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} = \frac{(Ze)^2E^2}{4p^4\sin^4(\theta/2)}, \tag{6.13}$$

where Ze is the charge on the target which we assume infinitely heavy so that there is no recoil, e is the charge on the incident particle, p and E are the momentum and energy of the incident beam, θ the scattering angle and $\alpha = e^2/4\pi = 1/137$ is the fine structure constant. We may note here that the scattering has no dependence on the size of the target, its mass or any other dynamical variable.

6.2.2 Mott Scattering

If we incorporate spin of the incident particle say electron and describe it by Dirac wave function then, an extra term $\cos^2(\theta/2)$ appears in the cross-section. We obtain the Mott formula for the scattering of relativistic spin 1/2 electron by spinless point like target,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{(Ze)^2\cos^2(\theta/2)}{4p_0^2\sin^4(\theta/2)[1+(2p_0/M)\sin^2(\theta/2)]}. \tag{6.14}$$

For small scattering angles, this is identical with the Rutherford formula. The additional term in the denominator accounts for the target (nuclear) recoil and is important only for high energy electrons and large scattering angles.

6.2.3 Form Factor

The cross-section for scattering due to an extended target, e.g. due to a particle with charge density $\rho(r)$, can be written from that of the point particle (6.13) by inclusion of a form factor $F(q^2)$:

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} [F(q^2)]^2. \tag{6.15}$$

Here, $\mathbf{q}$ is the momentum transfer from the projectile to the target, $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final momenta. The form factor $F(q^2)$ is simply the Fourier transform of the charge distribution $\rho(r)$ and is a function only of the magnitude $|\mathbf{q}|$ if ρ is spherically symmetric. For a point particle,

$$\rho(\mathbf{r}) = \delta(\mathbf{r}) \tag{6.16}$$

so that $F(q^2) = 1$. If we want to probe a charge distribution inside a hadron, we need

$$q^2 >> \frac{\hbar c}{\langle r \rangle} \approx \text{GeV}^2, \tag{6.17}$$

$\langle r \rangle$ being the hadronic (nuclear) size. We, therefore, cannot use simple non-relativistic approach and have to use relativistic methods.

An extended charge distribution in space leads to a decreasing and vanishing $F(q^2)$ as $q^2 \to \infty$ (see Table 6.1). Only a point-like target allows $F(q^2)$ = constant for $q^2 \to \infty$.

Table 6.1 Examples of Form Factors

	Charge distribution	*Form factor*	
Point	$\rho(r) = \delta(r - r_0)$	$F(q^2) = 1$	Unity
Exponential	$\rho(r) = \dfrac{a^3}{8\pi} e^{-ar}$	$F(q^2) = \left[\dfrac{1}{(1 + q^2/a^2)}\right]^2$	Dipole
Yukawa	$\rho(r) = \dfrac{a^2}{4\pi r} e^{-ar}$	$F(q^2) = \dfrac{1}{(1 + q^2/a^2)}$	Pole
Gaussian	$\rho(r) = \left[\dfrac{a^2}{2\pi}\right]^{3/2} e^{-(a^2 r^2/2)}$	$F(q^2) = e^{-(q^2/2a^2)}$	Gaussian

6.2.4 Currents in Electromagnetic Interactions

Consider an electromagnetic process (Fig. 6.1)

$$\alpha \to \beta + \gamma, \tag{6.18}$$

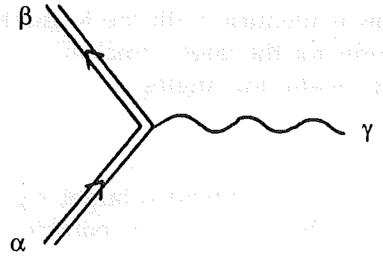


Fig. 6.1 Basic EM process.

where α and β are systems of one or more hadrons and γ is a photon of momentum $\hbar\mathbf{q}$. The transition amplitude to lowest order is given by

$$e\,\langle\beta|\,j_\mu^{\mathrm{EM}}\,|\alpha\rangle\,\varepsilon^\mu \tag{6.19}$$

where ε^μ is the photon polarization vector. The photon probes the structure of these states via the current operator j_μ^{EM}. For example, to lowest order the process

$$e + \alpha \to e + \beta$$

is described by the Feynman diagram (Fig. 6.2). The amplitude is given by

$$e^2\langle\beta|j_\mu^{\mathrm{EM}}|\alpha\rangle\frac{1}{q^2}\langle e_F|j^\mu|e_I\rangle. \tag{6.20}$$

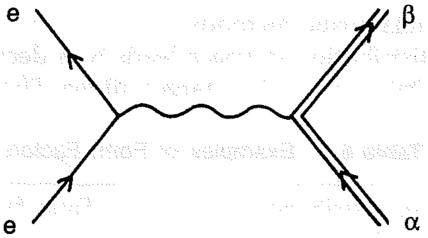


Fig. 6.2 Electron-proton scattering.

The matrix element of the electron part of the electromagnetic current is a trivially known factor

$$\langle e_F|j^\mu|e_I\rangle = \bar{u}(p_F)\gamma^\mu u(p_I). \tag{6.21}$$

Here, p_I and p_F are the initial and final electron momenta and the virtual photon has the momentum $q = p_I - p_F$. The photon here is off-mass shell, i.e. $q^2 \neq 0$, in general. The matrix element $\langle\beta|\,j_\mu^{\mathrm{EM}}\,|\,\alpha\,\rangle$ contains all the complexities of the strong interactions. For scattering of electrons on protons we must take the following factors into account:

1. Electrons and protons are spin 1/2 particles and proton magnetic moment contributes to the scattering.
2. Proton has a finite mass and so can recoil.
3. Proton is not a point object and has a charge distribution.

6.2.5 Nucleon Form Factors

The scattering of electrons from a nucleon (proton or neutron) due to the electromagnetic interaction is represented by the Feynman diagram (Fig. 6.3), with α and β being replaced by proton. We write the matrix element for the process as

$$e^2 j_e^\mu(p', p) \frac{1}{q^2} \langle N(P')|J_\mu(0)|N(P)\rangle \tag{6.22}$$

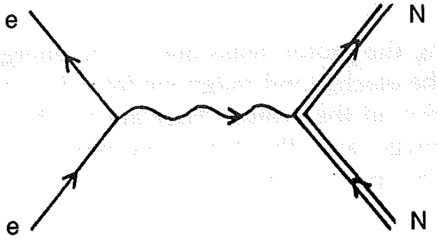


Fig. 6.3 Electron-nucleon elastic scattering.

The interpretation of this expression is as follows: the factor $1/q^2$ represents the propagation of a virtual photon of four momentum q between electron and the nucleon, where q is the recoil momentum (momentum transfer)

$$q_\mu = P'_\mu - P_\mu = p_\mu - p'_\mu . \tag{6.23}$$

The factor j_e^μ is the electron's four current density which, assuming no internal structure, is given by

$$j_e^\mu(p', p) = \bar{u}(p')\gamma^\mu u(p), \tag{6.24}$$

where u, $\bar{u}$ are Dirac spinors for the electron. The current density for the nucleon J^μ will include all the effects of the internal structure. We may write

$$\langle p(P')\,|\,J^\mu(0)\,|\,p(P)\rangle = [F_1(q^2)\gamma^\mu + i\frac{\kappa}{2M} F_2(q^2)\sigma^{\mu\nu}q_\nu], \tag{6.25}$$

where F_1 and F_2 are respectively the Dirac and Pauli form factors with $F_1(0) = 1$ and $\kappa F_2(0)$ being the anomalous magnetic moment. So $F_2(0) = 1$. The assumptions implied in (6.25) are:

1. Relativistic covariance, which means that J_μ is a four vector
2. Differential law of current conservation

$$\partial^\mu J_\mu = 0 \tag{6.26}$$

   which in momentum space reads

$$(P' - P)^\mu J_\mu(P', P) = 0 \tag{6.27}$$

3. That nucleon is a Dirac particle.

Exercise Deduce (6.25).

The functions $F_{1,2}^{p,n}(q^2)$ describe the internal structure of proton (neutron) and in the static limit $q^2 \to 0$ take the value unity (zero). F_1 and F_2 are the relativistic generalizations of the form factor $F(q^2)$ defined earlier in (6.15). When these points are taken into account, the cross-section of scattering of electrons on protons can be represented in

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \frac{E'}{E}\left[\frac{G_E^2 + tG_M^2}{1+t}\cos^2(\theta/2) + 2tG_M^2 \sin^2(\theta/2)\right], \tag{6.28}$$

where $t = -q^2/4M^2$, M being the proton mass and E' the energy of the projectile after the scattering. G_E and G_M are the electric and magnetic form factors related to the charge and magnetic moment distribution in the proton. They are functions of q^2 and are normalized at $q^2 = 0$ to the total charge and the total magnetic moment of the proton, i.e. $G_E^{\text{proton}}(0) = 1$, $G_M^{\text{proton}}(0) = 2.79$. We can also write (6.28) in the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} [F_1^2 + t\{2(F_1 + \kappa F_2)^2 \tan^2(\theta/2) + \kappa^2 F_2^2\}], \tag{6.29}$$

where F_1 and F_2 are the Dirac and Pauli form factors defined in (6.25); they are also functions of q^2. G_E and G_M are related to F_1 and F_2 through

$$G_E = F_1 + t\kappa F_2, \tag{6.30}$$

$$G_M = F_1 + \kappa F_2. \tag{6.31}$$

G_E and G_M are the Fourier transforms of the charge and magnetic distributions. If we assume both electron and proton point-like with magnetic moments $e\hbar/(2mc)$ and $e\hbar/(2Mc)$ respectively, where m and M are the electron and proton masses, we will have, from (6.28), the relation

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2[\cos^2(\theta/2) + (q^2/2M^2)\sin^2(\theta/2)]}{4p_0^2 \sin^4(\theta/2)\,[1 + (2p_0/M)\sin^2(\theta/2)]}. \tag{6.32}$$

The factor in square brackets in the denominator represents the recoil correction. This expression differs from the Mott formula in the inclusion of the second factor in the parentheses in the numerator due to the magnetic scattering. Large angle and high momentum transfer correspond to the close collisions where magnetic scattering dominates (recall that the magnetic dipole potential varies as $1/r^2$ compared to $1/r$ for electric potential). The cross-section can also be represented as

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} [A + B\tan^2(\theta/2)], \tag{6.33}$$

where A and B are related to the Dirac (charge) and Pauli (anomalous magnetic moment) form factors by

$$A = F_1^2 + t\kappa^2 F_2^2, \qquad B = 2t[F_1 + \kappa F_2], \tag{6.34}$$

where κ is the anomalous magnetic moment of proton (in nuclear magneton). The final result is the so-called Rosenbluth cross-section. Measuring the differential cross-section as a function of the angle of scattering, we obtain A and B as functions of q^2, and hence, the form factors F_1 and F_2. The first experiments measuring the form factors of the nucleon and thereby pointing out the deviation of the scattering from a point particle and indicating the structure of proton and neutron were performed by R. Hofstader and his collaborators in 1961 at Stanford and have since been carried out in many laboratories. The experiments show that the form factors obey the scaling law

$$G_E^p(q^2) = \frac{G_M^p(q^2)}{|\mu_p|} = \frac{G_M^n(q^2)}{|\mu_n|} = G(q^2), \tag{6.35}$$

$$G_E^n = 0, \tag{6.36}$$

with

$$G(q^2) = \left(1 + \frac{Q^2}{M_v^2}\right)^{-2}, \tag{6.37}$$

where $Q^2 = -q^2$. This behaviour corresponds to a charge distribution of the type

$$\rho(r) = e^{-M_v r} \tag{6.38}$$

with $M_v = 0.84$ GeV and $R_{\mathrm{rms}} = \sqrt{12}/M_v \approx 0.80\, fm$. This fit is good upto $Q^2 \approx 25$ GeV^2. Since $(d\sigma/d\Omega)$ drops rapidly like $(1/Q^2)$ as Q increases, we see that the elastic cross-section dies out very quickly as one moves to large Q^2. The remarkable discovery by W.K.H. Panofsky in 1968 that for inelastic scattering, the form factors do not decrease at large Q^2, is the basis for the idea that nucleon has parton structure.

6.3 The Quark Parton Model

6.3.1 Inelastic Scattering: The kinematics

Let us consider some kinematical features common to lepton-nucleon scattering experiments—elastic or inelastic. Let us consider the process

$$e + N \rightarrow e + X\backslash \tag{6.39}$$

which occurs predominantly through the exchange of a single virtual photon as depicted in Fig. 6.4. Let us denote by $p = (E, \mathbf{p})$ and $p' = (E', \mathbf{p}')$ the four momenta of the initial and final leptons. The four momentum of the initial nucleon is labelled by $P = (E, \mathbf{p})$, and this reduces to $(M, 0)$ in the laboratory frame. All the leptons are point-like Dirac particles. A proton (neutron), on the other hand, has an internal structure of its own. This is shown by a blob at the proton vertex. As for the final hadronic state, it may consist of one nucleon as in the elastic scattering or two or more hadrons as in the inelastic case. We denote its total four momentum by P' and its invariant mass by W. In the deep inelastic scattering, a large amount of energy and momentum are transferred to the target proton, breaking it

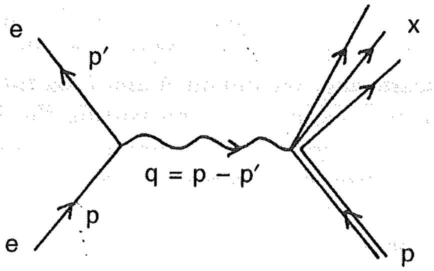


Fig. 6.4 Inelastic electron-proton scattering.

up into hadron X. Because of the large momentum transfer squared carried by the photon, the wavelength of photon is small and so the interaction probes the short range distance structure of the proton target. The energy transfer from the initial lepton to the hadronic system is

$$\nu = q_0 = E - E', \tag{6.40}$$

and the four-momentum transfer is

$$-q = p' - p = P - P'. \tag{6.41}$$

Thus, we have

$$q^2 = (p' - p)^2 = p'^2 + p^2 - 2p \cdot p'. \tag{6.42}$$

Neglecting the lepton mass, this becomes

$$\begin{aligned} q^2 &= -2p \cdot p' = -Q^2 = 2\,(\mathbf{p} \cdot \mathbf{p}' - EE') \\ &= -2EE'(1 - \cos\theta) = -4EE' \sin^2(\theta/2), \end{aligned} \tag{6.43}$$

where θ is the lepton scattering angle. If W is the effective mass of the hadronic system, we also have

$$q^2 = (P - P')^2 = M^2 + W^2 - 2P \cdot P' \tag{6.44}$$

which in the lab frame ($\mathbf{P} = 0$) becomes

$$q^2_{\text{lab}} = W^2 - M^2 - 2M\nu, \tag{6.45}$$

because in the lab frame $2P \cdot P' = 2M(M + \nu) = 2M^2 + 2M\nu$. For the elastic scattering, q^2 and ν are not independent since $M = W$ and $q^2 = -2M\nu$. Now the central point is as follows: In an inelastic process, there are two independent variables, the energy loss and the momentum transfer. So, in general, the inclusive scattering cross-section depends upon two variables, viz.

$$q^2, \quad \nu = q_0 = E - E' = \frac{P \cdot q}{M}. \tag{6.46}$$

But as we shall see later, the deep inelastic scattering depends only on one variable indicating that it may be due to superposition of elastic scattering. Let us define a dimensionless variable

$$x = -\frac{q^2}{2P\cdot q}. \tag{6.47}$$

For the elastic scattering, $q^2 = -2P \cdot q$ so that $x = 1$. For the inelastic scattering,

$$x = 1 - \frac{W^2 - M^2}{2P\cdot q}, \tag{6.48}$$

where

$$W^2 = P'^2 = (P + q)^2 = P^2 + q^2 + 2P \cdot q = M^2 + q^2 + 2P \cdot q. \tag{6.49}$$

The cross-section measured in deep inelastic lepton scattering is an inclusive cross-section. One does not observe the final hadrons but merely the final lepton energy and scattering angle. There may be any number of hadrons emerging and any distribution of energy amongst the hadrons. As in elastic scattering, we formulate a scheme that separates the known from the unknown. We know how to handle the lepton vertex. We parametrize the hadronic part by imposing the known constraints. The amplitude for the inelastic electron scattering can be written as

$$A \sim \bar{u}(p')\gamma^\mu u(p)\frac{1}{q^2}\langle X|J_\mu|p\rangle, \tag{6.50}$$

where X is any hadron state. The spin averaged square matrix element has the form

$$\frac{1}{4}\sum_{\text{spin}} |A|^2 = L^{\mu\nu}W_{\mu\nu}, \tag{6.51}$$

where

$$\left.\begin{aligned} L^{\mu\nu} &= \frac{1}{2}\Sigma_{s'}\bar{u}(p', s')\,\gamma^\mu\, u(p, s)\;\bar{u}(p, s)\,\gamma^\nu u(p's'), \\ W_{\mu\nu} &= \frac{1}{2}\Sigma_X \langle p|\, J_\mu^\dagger \,|X\rangle\,\langle X|J_\nu|p\rangle\,(2\pi)^3\delta(P + q - P_X). \end{aligned}\right\} \tag{6.52}$$

$L^{\mu\nu}$ can be evaluated to yield

$$L^{\mu\nu} = 2[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' - m^2)]. \tag{6.53}$$

$L^{\mu\nu}$ is symmetric in μ, ν. So $W_{\mu\nu}$ will also be symmetric in μ, ν. Using the Lorentz covariance and the conservation of electromagnetic current, we can write

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(Q^2, \mu) + \left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2}q_\nu\right)\frac{1}{M^2}W_2(Q^2, \nu). \tag{6.54}$$

Exercise Prove (6.54).

Solution Since $W_{\mu\nu}$ is symmetric with respect to μ and ν, we can, in general, write

$$W_{\mu\nu} = -g_{\mu\nu}W_1 + (P_\mu P_\nu/M^2)W_2 + (1/M^2)\,(P_\mu q_\nu + P_\nu q_\mu)W_4 + (q_\mu q_\nu/M^2)W_5.$$

Conservation of charge requires

$$q^{\mu} W_{\mu\nu} = 0 = W_{\mu\nu} q^{\nu},$$

giving us

$$-q_{\nu} W_1 + (q \cdot P/M^2)\, P_{\nu} W_2 + (1/M^2)\,(q \cdot P)\, q_{\nu}\, W_4 + (q^2/M^2) P_{\nu} W_4 + (q^2 q_{\nu}/M^2) W_5 = 0.$$

Now, since P and q are arbitrary, their coefficients should separately be zero. Thus, we get

$$W_4 = -\,(q \cdot P/q^2) W_2,$$

$$W_5 = (M^2/q^2) W_1 + ((q \cdot P)^2/q^4) W_2.$$

Substituting these equations in the general expression for $W_{\mu\nu}$, we obtain (6.54).

The unpolarized differential cross-section for the energy and angular distribution of scattered electron (lepton), with the final hadrons unobserved, is then given by

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} [W_2 \cos^2(\theta/2) + 2W_1 \sin^2(\theta/2)]. \tag{6.55}$$

Thus, the inelastic electron-proton scattering differential cross-section can be written entirely in terms of two independent target structure functions W_1 and W_2 which are relativistically invariant, depending upon the Lorentz invariant variables q^2 and ν. They carry all the information on the target structure relevant to the lepton target scattering and thus hide all our ignorance of the hadronic structure the same way the form factors did in the case of elastic scattering. A look at the Rosenbluth formula shows that for the elastic scattering ($W^2 = P'^2 = M^2$) the structure functions are of the form

$$W_1^{\text{el}}(\nu, q^2) = \frac{Q^2}{4M^2} G_{\text{M}}(Q^2)\, \delta\left(\nu - \frac{Q^2}{2M}\right), \tag{6.56}$$

$$W_2^{\text{el}}(\nu, q^2) = \frac{G_{\text{E}}(Q^2) + \dfrac{Q^2}{4M^2} G_{\text{M}}(Q^2)}{1 + \dfrac{Q^2}{4M^2}}\, \delta\left(\nu - \frac{Q^2}{2M}\right). \tag{6.57}$$

We have obtained the above form by comparing the inelastic scattering formula with the point Rosenbluth formula along with the condition

$$\nu = -\frac{q^2}{2M} = \frac{Q^2}{2M} \tag{6.58}$$

For the elastic scattering from point particle of unit charge, we have

$$W_1^{\text{point}}(\nu, q^2) = \frac{Q^2}{4M^2}\, \delta\left(\nu - \frac{Q^2}{2M}\right), \tag{6.59}$$

$$W_2^{\text{point}}(\nu, q^2) = \delta\left(\nu - \frac{Q^2}{2M}\right), \tag{6.60}$$

because $G_E(Q^2) = G_M(Q^2) = 1$ for the point target. The elastic scattering cross-section due to a point target with charge e_i is given by

$$\frac{d^2\sigma}{dQ^2\,d\nu} = \frac{\pi\alpha^2}{4p^2\sin^4(\theta/2)}\,\frac{1}{pp'}\,\{e_i^2\cos^2(\theta/2) + e_i^2\frac{Q^2}{4M_i^2}\,2\sin^2(\theta/2)\}\,\delta\left(\nu - \frac{Q^2}{2M_i}\right). \tag{6.61}$$

J.D. Bjorken conjectured that for large ν and $Q^2 = -q^2$, W_1 and νW_2 become functions of the dimensionless ratio x:

$$MW_1(\nu, Q^2) = F_1(x), \tag{6.62}$$

$$\nu W_2(\nu, Q^2) = F_2(x). \tag{6.63}$$

This scaling phenomenon was indeed observed by J. Friedman and H. Kendall in very early SLAC data in 1972. The inelastic scattering indicates that $q^2(d\sigma/dq^2)$ is very mildly q^2 dependent and has Mott value in contrast to form factor falling off with Q^2 characteristic of hadronic size. This lack of pronounced q^2 dependence suggests that the target (proton) constituents are pointlike. A simple approach to the understanding of this scaling phenomenon is offered in the naive parton model due to R.P. Feynman in 1969. In this model the nucleon is assumed to consist of point like constituent partons with well-defined quantum numbers.

Infinite momentum frame

P is large so that mass and transverse momentum can be neglected. Let ξP be the momentum of one of the partons. It is scattered by absorbing the momentum transfer q with momentum $\xi P + q$; the rest mass of the parton m_q can then be neglected. Thus,

$$(\xi P + q)^2 = m_q^2 = 0,$$

$$\xi^2 P^2 + q^2 + 2\xi P \cdot q = 0.$$

If

$$\xi^2 P^2 = \xi^2 M^2 << q^2,$$

then we obtain

$$\xi = -\frac{q^2}{2P\cdot q}. \tag{6.64}$$

This is a Lorentz invariant quantity and so may be evaluated in any frame. In the lab (target rest) frame $P \cdot q = M\nu$; so

$$\xi = -\frac{q^2}{2M\nu}. \tag{6.65}$$

For elastic scattering from a parton of mass m_q at rest, $q^2 = -2m_q\nu$. Therefore, we may write

$$\xi = -\frac{q^2}{2M\nu} = x = \frac{m_q}{M}. \tag{6.66}$$

So x is the fractional mass of the nucleon carried by a free parton initially at rest in the laboratory frame. This fraction x may take any value from 0 and 1. Let the probability that the ith parton carries a fraction of momentum between x and $x + dx$ be $f_i(x)dx$. Then

$$\sum_i \int d\xi \xi f_i(\xi) = 1. \tag{6.67}$$

This comes from the parton model postulate

$$\frac{d^2\sigma}{d\Omega\, dE'}_{ep \to eX} = \sum_i e_i^2 \int d\xi f_i(\xi) \left(\frac{d^2\sigma}{d\Omega\, dE'} \right)_{i\text{th parton}} \tag{6.68}$$

Equation (6.67) states that all the fractions add up to give unity, and no momentum is lost. However, for the struck parton x may be identified with the Bjorken variable. For elastic scattering from the parton of mass m_q and charge e_q, we can write

$$W_1^q(\nu, Q^2) = e_q^2 \frac{Q^2}{4M^2\xi^2} \delta\left(\nu - \frac{Q^2}{2M\xi} \right), \tag{6.69}$$

$$W_2^q(\nu, Q^2) = e_q^2 \delta\left(\nu - \frac{Q^2}{2M\xi} \right). \tag{6.70}$$

With the help of the property

$$\delta(g(x)) = \frac{\delta(x - x_0)}{g'(x)\big|_{x=x_0}} \tag{6.71}$$

of the Dirac delta function (where $g(x_0) = 0$), we can write

$$\delta\left(\nu - \frac{Q^2}{2M\xi} \right) = \frac{2Mx^2}{Q^2} \delta(\xi - x) = \frac{x}{\nu} \delta(\xi - x), \tag{6.72}$$

giving

$$MW_1^q(\nu, Q^2) = e_q^2 \frac{x^2}{2\xi^2} \delta(\xi - x), \tag{6.73}$$

$$\nu W_2^q(\nu, Q^2) = e_q^2 x \delta(\xi - x). \tag{6.74}$$

The structure functions of the nucleon are obtained by an incoherent sum over the partons appropriately weighted by the momentum distribution $f_i(\xi)$.

$$\begin{aligned} F_2^N = \nu W_2^N &= \int d\xi \sum_i e_i^2 f_i(\xi) x \delta(\xi - x) \\ &= \int d\xi \sum_i e_i^2 f_i(\xi) \xi \delta(\xi - x) = \sum_i e_i^2 x f_i(x). \end{aligned} \tag{6.75}$$

Similarly,

$$F_1^N = MW_1^N = \int d\xi \sum_i e_i^2 f_i(\xi) \frac{x^2}{2\xi^2}\, \delta(\xi - x)$$

$$= \int d\xi \sum_i e_i^2 f_i(\xi) \frac{x^2}{2\xi^2}\, \delta(\xi - x) = \frac{1}{2} \sum_i e_i^2 f_i(x). \tag{6.76}$$

So,

$$2xF_1(x) = F_2(x). \tag{6.77}$$

This is the Callan-Gross relation valid for spin $\frac{1}{2}$ parton.

6.3.2 Spin of the Parton

The data show that, except possibly for low x and q^2, there is good agreement with the Callan-Gross relation. For spin 0 partons,

$$2xF_1/F_2 = 0 \tag{6.78}$$

which is excluded by the experimental data.

Alternative derivation of Callan-Gross relation. Inclusive electron-nucleon scattering is described by

$$\frac{d^2\sigma}{dq^2} = \frac{4\pi\alpha^2 E'}{q^4 EM} [W_2 \cos^2(\theta/2) + 2W_1 \sin^2(\theta/2)]. \tag{6.79}$$

The formula for the elastic scattering of an electron on a point particle with spin $\frac{1}{2}$, mass m and charge Ze, with $g = 2$ given by (6.61). By comparing the angular parts, we get

$$\frac{W_2}{2W_1} = \frac{2m^2}{q^2}. \tag{6.80}$$

Using $x = m/M = q^2/(2M\nu)$, and defining $(\nu/M)\, W_2 = F_2$ and $W_1 = F_1$, we find that

$$2xF_1(x) = F_2(x). \tag{6.81}$$

This is the Callan-Gross relation.

6.3.3 Structure Functions

One of the most interesting results emerging from the study of inelastic lepton-hadron scattering at high energies and large momentum transfers is the possibility of obtaining detailed information about the fundamental constituents of hadrons. The large q^2 short wavelength virtual photon, prepared by the inelastic scattering probes the proton, revealing its constituents named partons by Feynman. Let us identify partons with the Gell-Mann Zweig quarks. In this model the bare electromagnetic current is

$$j_\mu^{\text{EM}} = \frac{2}{3}\, \bar{u}(x)\, \gamma_\mu u(x) - \frac{1}{3}\, \bar{d}(x) \gamma_\mu d(x) - \frac{1}{3} \bar{s}(x) \gamma_\mu s(x) \ldots, \tag{6.82}$$

where $u(x)$, $d(x)$, $s(x)$... stand for the fields of u, d, s quarks. The ellipsis corresponds to charm, b and t quarks. Similarly, we can write the weak current.

We now consider the structure functions for the elementary lepton quark processes. For simplicity we keep only three flavours of quarks omitting heavy quark contributions which are suppressed due to higher threshold. It follows that the e-p structure function is

$$F_2^{ep}(x) = x\left\{\frac{4}{9}\,[u^p(x) + \bar{u}^p(x)] + \frac{1}{9}\,[d^p(x) + \bar{d}^p(x)] + \frac{1}{9}[s^p(x) + \bar{s}^p(x)]\right\}, \tag{6.83}$$

where $u(x)$ is the momentum density distribution of quark of type u. (Remember that $u(x)\,dx$ is the number of quarks of type u with momentum fraction between x and $x + dx$.) Similarly, $\bar{u}(x)$ represents the momentum distribution for $\bar{u}$ and so for others. $u(x)$ is the probability distribution of finding an up quark in the proton with fraction of longitudinal momentum x. The factors 4/9 and 1/9 are the squares of the charges of the appropriate quarks. We can write similar expression for the neutron.

$$F_2^{en}(x) = x\left\{\frac{4}{9}\,[u^n(x) + \bar{u}^n(x)] + \frac{1}{9}\,[d^n(x) + \bar{d}^n(x)] + \frac{1}{9}[s^n(x) + \bar{s}^n(x)]\right\}. \tag{6.84}$$

Since the number of u quarks in a proton is the same as the number of d quarks in a neutron, from isospin symmetry,

$$u^p(x) = d^n(x) = u(x),$$
$$d^p(x) = u^n(x) = d(x),$$
$$s^p(x) = s^n(x) = s(x).$$

Therefore,

$$\left.\begin{aligned} F_2^{ep}(x) &= \left\{\frac{x}{9}\,[4(u(x) + \bar{u}(x)) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]\right\}, \\ F_2^{en}(x) &= \left\{\frac{x}{9}\,[(u(x) + \bar{u}(x)) + 4(d(x) + \bar{d}(x)) + s(x) + \bar{s}(x)]\right\}. \end{aligned}\right\} \tag{6.85}$$

The structure functions for the neutrino and antineutrino nucleon inelastic scattering can also be written. In proton the average momentum carried by the up quark would be twice the average carried by the down quark since there are twice as many of them:

$$\int_0^1 xu(x)\,dx = 2\int_0^1 xd(x)\,dx. \tag{6.86}$$

Combining the two, we get

$$\int_0^1 xd(x)\,dx = \int_0^1 F_2(x)\,dx. \tag{6.87}$$

Experimentally,

$$\left.\begin{aligned} \int_0^1 xd(x)\,dx &= 0.18, \\ \int_0^1 xu(x)\,dx &= 0.36. \end{aligned}\right\} \tag{6.88}$$

The average fraction of momentum carried by the quarks is then 0.54. The remaining momentum is expected to be carried by gluons.

Further predictions of the model can be made by making some special assumptions. We assume that the nucleon consists of three valence quarks u_v, u_v, d_v (or d_v, d_v, u_v). The quark-antiquark pairs $u_s \bar{u}_s$, $d_s \bar{d}_s$, $s_s \bar{s}_s$ are produced by the gluons emitted by these valence quarks. These are called 'sea' quarks. As a first approximation, we may assume that only three lightest flavour quarks (u, d, s) are present in the 'sea'. Hence,

$$\begin{aligned} u(x) &= u_v(x) + u_s(x), \\ d(x) &= d_v(x) + d_s(x). \end{aligned} \tag{6.89}$$

In the limit when all the quarks are massless, the probability for a gluon to produce $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ pairs would be the same. Thus in this limit we can make the assumption that the 'sea' quarks are present in equal proportion:

$$u_s(x) = \bar{u}_s(x) = d_s(x) = \bar{d}_s(x) = s_s(x) = \bar{s}_s(x) = s(x). \tag{6.90}$$

Since the net number of u quarks in the proton is 2, it follows that

$$\int_0^1 dx\, [u(x) - \bar{u}(x)] = 2. \tag{6.91}$$

Similarly, for d and s quarks we require that

$$\int_0^1 dx\, [d(x) - \bar{d}(x)] = 1, \tag{6.92}$$

$$\int_0^1 dx\, [s(x) - \bar{s}(x)] = 0. \tag{6.93}$$

These equations which express the overall quark content of the proton provide sum rules which can be tested experimentally. In terms of the valence quark and sea quark distribution,

$$\frac{1}{x} F^{\text{ep}} = \frac{1}{9} [4u_v + d_v] + \frac{4}{3} s, \tag{6.94}$$

$$\frac{1}{x} F_2^{\text{en}} = - \frac{1}{9} [u_v + 4d_v] + \frac{4}{3} s, \tag{6.95}$$

where 4/3 is the sum over the six sea quark distribution

$$\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) \times 2 = \frac{4}{3}. \tag{6.96}$$

We now define the ratio

$$\gamma(x) = \frac{F_2^{\text{en}}}{F_2^{\text{ep}}} = \frac{4d_v(x) + u_v(x) + 12s(x)}{4u_v(x) + d_v(x) + 12s(x)}. \tag{6.97}$$

We then have, for any x,

$$\gamma(x) \approx 1 \qquad \text{if } s(x) \text{ dominates,} \tag{6.98}$$

$$\gamma(x) \approx 1/4 \qquad \text{if } u_v(x) \text{ dominates,} \tag{6.99}$$

$$\gamma(x) \approx 4 \qquad \text{if } d_v(x) \text{ dominates,} \tag{6.100}$$

$$\gamma(x) \approx 2/3 \qquad \text{if } u_v(x) = 2d_v(x),\ s(x) = 0. \tag{6.101}$$

In general, $1/4 \le \gamma(x) \le 4$. The data shows that the ratio $\gamma(x)$ varies between 1 at $x = 0$ and 0.2 at $x = 1$. Therefore, we conclude that

1. the small x is dominated by the sea quarks, whereas
2. the large x is dominated by $u_v(x)$; $d_v(x)$ does not dominate anywhere in the x-domain.

The first of these conclusions is easy to understand: the quark-antiquark pairs are created by the gluons emitted by the valence quarks, in our picture. Obviously, the momentum fraction of valence quarks is now shared by two sea quarks. These sea quarks can in turn emit gluons which the pair produce. In the limit of massless quarks, this process can go on ad-infinitum. Therefore, as the number of sea quarks increases, their share of hadron momentum also goes down.

An indirect evidence for the presence of the gluons is provided by the momentum sum rule. The gluons do not couple to the leptonic current of electrons or neutrinos. However, if they are present in the hadron, they must carry a fraction of the hadron momentum. Integrating over the experimental data on $F_2^{\text{ep, en}}(x)$, we obtain

$$\begin{aligned}\int F_2^{\text{ep}}(x)\,dx &= \int dx \left\{\frac{x}{9}\,[4(u(x) + \bar{u}(x)) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]\right\}\\ &= \frac{4}{9}\,\varepsilon_u + \frac{1}{9}\,\varepsilon_d + \frac{1}{9}\,\varepsilon_s,\end{aligned} \tag{6.102}$$

$$\begin{aligned}\int F_2^{\text{en}}(x)\,dx &= \int dx \left\{\frac{x}{9}\,[(u(x) + \bar{u}(x)) + 4(d(x) + \bar{d}(x)) + s(x) + \bar{s}(x)]\right\}\\ &= \frac{1}{9}\,\varepsilon_u + \frac{4}{9}\,\varepsilon_d + \frac{1}{9}\,\varepsilon_s,\end{aligned} \tag{6.103}$$

where ε_q is the fraction of the momentum carried by the quark and antiquark of a given flavour. We also have the constraint that the sum over all the constituent momentum fractions should be one, so that $\varepsilon_u + \varepsilon_d + \varepsilon_s = 1$. We then obtain $\varepsilon_u \approx 0.24$, $\varepsilon_d \approx 0$, $\varepsilon_s \approx 0.76$, which does not appear reasonable, as strange quarks dominate over all other flavours. If we introduce gluons, the constraint becomes $\varepsilon_u + \varepsilon_d + \varepsilon_s + \varepsilon_g = 1$. Now we have four unknowns and only three numbers. If we now neglect strange quarks (they only contribute at small x and their total integrated fraction may be small), then we get

$$\varepsilon_u \approx 0.36, \qquad \varepsilon_d \approx 0.18, \qquad \varepsilon_g \approx 0.46. \tag{6.104}$$

In this picture then gluons carry nearly 50% of the hadron momentum. An independent evidence for the momentum sum rule is provided by the neutrino structure functions which will be considered later.

6.3.4 Neutrino-Nucleon (CC) Reactions

Unlike the electron scattering, neutrino proton ($\nu_\mu - p$) scattering proceeds through (Fig. 6.5) weak boson exchanges, $W^\pm$ and Z. The structure of the lepton vertex also changes since neutrino scattering involves weak current. The amplitude for the process can be written as

$$A \sim \bar{u}_\mu(p')\gamma^\mu(1 + \gamma_5)u_\nu(p)\,\langle X | J_\mu | p\rangle, \tag{6.105}$$

$$|A|^2 \sim L^{\mu\nu}W_{\mu\nu}. \tag{6.106}$$

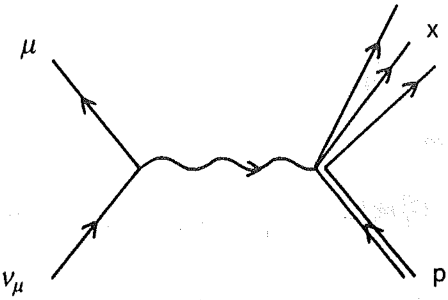


Fig. 6.5 Inelastic neutrino-proton scattering.

Because of parity violation in the weak interaction, we would have an additional term in the lepton tensor $L^{\mu\nu}$ as well as an additional structure function, and hence an additional term in the hadron tensor $W_{\mu\nu}$.

$$L^{\mu\nu} = p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\,.\,p' - m^2) \pm i\varepsilon^{\mu\nu\lambda\sigma}p_\lambda p_\sigma. \tag{6.107}$$

The + sign is associated with the left-handed neutrino and – sign with the right-handed antineutrino projectile.

$$W_{\mu\nu} = -g_{\mu\nu}W_1 + \frac{1}{M^2}P_\mu P_\nu W_2 - \frac{i}{2M}\varepsilon_{\mu\nu\gamma\delta}P^\gamma P^\delta W_3 \tag{6.108}$$

For the charged current inclusive reaction, $\nu_\mu p \to \mu^- X^{++}$, we obtain the following cross-section formula:

$$\begin{aligned}\frac{d^2\sigma^{\nu p}}{dQ^2 d\nu} = \frac{G_F^2}{2\pi m_p^2}\frac{E_\mu}{E_\nu}\{&\cos^2(\theta/2)\,W_2(Q^2,\nu) + 2\sin^2(\theta/2)\,W_1(Q^2,\nu)\\ &\pm \frac{E_\nu + E_\mu}{m_p}\sin^2(\theta/2)\,W_3(Q^2,\nu)\}.\end{aligned} \tag{6.109}$$

The + sign is for νp scattering while the – sign is for $\bar{\nu}p$. The cross-section is written in terms of three-structure functions. They correspond in ep scattering to: $W_1 \to$ magnetic scattering, $W_2 \to$ electric scattering, and $W_3 \to$ parity violating term due to V-A interference. In terms of the variables x and $y = \nu/E_\nu$, the cross-section can be represented as:

$$\frac{d^2\sigma^{\nu,\bar{\nu}}}{dx\,dy} = \frac{G_F^2 m_p E_\nu}{\pi}\left\{\left(1 - y - \frac{m_p xy}{2E_\nu}\frac{\nu}{m_p}\right)W_2 + xy^2 W_1 \pm xy\left(1 - \frac{y}{2}\right)\frac{\nu}{m_p}W_3\right\}. \tag{6.110}$$

For $E_\nu >> m_p$, the term $m_p xy/(2E_\nu)$ can be neglected.

Bjorken scaling

Bjorken has shown that in the limit of $Q^2 \to \infty$, $\nu \to \infty$, the structure functions become functions of the variable x only:

$$W_1(Q^2, \nu) \to F_1(x), \tag{6.111}$$

$$\frac{\nu}{m_p} W_2(Q^2, \nu) \to F_2(x), \tag{6.112}$$

$$\frac{\nu}{m_p} W_3(Q^2, \nu) \to F_3(x). \tag{6.113}$$

The cross-section then becomes

$$\frac{d^2\sigma^{\nu,\bar{\nu}}}{dx\,dy} = \frac{G_F^2 m_p E_\nu}{\pi}\left\{(1-y)\,F_2(x) + y^2 F_1 \pm y\left(1-\frac{y}{2}\right)xF_3\right\}. \tag{6.114}$$

As a consequence of the Bjorken scaling, therefore, we have

$$\sigma \sim E_\nu, \qquad \frac{d^2\sigma}{dx\,dy} \sim E_\nu \text{ [term independent of } E_\nu]. \tag{6.115}$$

If we use the Callan-Gross relation, we obtain the following cross-section formula:

$$\frac{d^2\sigma^{\nu,\bar{\nu}}}{dx\,dy} = \frac{G_F^2 m_p E_\nu}{2\pi}\{[F_2(x) \pm xF_3]\,(1-y)^2 + [F_2(x) \mp xF_3(x)]\}. \tag{6.116}$$

We have two structure functions for each of the four scattering processes $\nu_\mu p$, $\nu_\mu n$, $\bar{\nu}_\mu p$, $\bar{\nu}_\mu n$ (8 in total; there are 12 if Callan-Gross relation is not valid). From flavour conserving charges, weak currents and from proton-neutron isospin symmetry, we have

$$F_i(\nu\, n) = F_i(\bar{\nu} p), \qquad F_i(\nu p) = F_i(\bar{\nu} n), \tag{6.117}$$

for $i = 1, 2$. Relations (6.117) are exact for F_2, and only approximate for F_3 (one has to neglect the strange sea quarks in the nucleon). One is thus left with four independent structure functions, for instance, $F_2(\nu p)$, $F_3(\nu p)$, $F_2(\bar{\nu} p)$, $F_3(\bar{\nu} p)$. But it is worthwhile to check the validity of charge symmetry.

The structure of (6.116) is of the type

$$\frac{d^2\sigma}{dx\,dy} = A(x)\,(1-y)^2 + B(x). \tag{6.118}$$

Thus fixing an x range and fitting the resulting y-distribution allows us to obtain the values for A and B. From these values then we may compute F_2 and F_3 from the relations for $\bar{\nu} p$, $A \sim F_2 - xF_3$, $B \sim F_2 + xF_3$. The procedure can be repeated for various x values, thus obtaining the complete dependences on x for F_2 and xF_3.

Let us now restrict our attention to isoscalar targets (with nuclei having the same number of protons and neutrons). The quark parton model predictions can be obtained by considering the relevant subprocesses, i.e.

$$\nu_\mu \begin{bmatrix} \bar{u} \\ d \\ s \end{bmatrix} \to \mu^- \begin{bmatrix} \bar{d} \\ u \\ u \end{bmatrix}, \tag{6.119}$$

$$\bar{\nu}_\mu \begin{bmatrix} u \\ \bar{d} \\ \bar{s} \end{bmatrix} \to \mu^+ \begin{bmatrix} d \\ \bar{u} \\ \bar{u} \end{bmatrix}. \tag{6.120}$$

The transitions involving $s \to u$ ($\bar{s} \to \bar{u}$) are Cabibbo suppressed. The individual subprocess cross-sections can be obtained using the form of the leptonic tensor for $V - A$ current and an analogous form for the quarks. The subprocess cross-section is given by

$$\frac{d\hat{\sigma}}{dy}(\nu_\mu d \to \mu^- u) = \frac{d\hat{\sigma}}{dy}(\nu_\mu d \to \mu^- u) = \frac{G^2}{\pi} xs, \tag{6.121}$$

$$\frac{d\hat{\sigma}}{dy}(\bar{\nu}_\mu u \to \mu^+ d) = \frac{d\hat{\sigma}}{dy}(\nu_\mu \bar{u} \to \mu^- \bar{d}) = \frac{G^2}{\pi} xs\,(1-y)^2. \tag{6.122}$$

Note that the νq scattering is isotropic, whereas $\nu\bar{q}$ is not. We now use the parton model ansatz

$$\frac{d\sigma}{dx\,dy}(\nu N \to \mu^- X) = \sum_i f_i(x)\,\frac{d\hat{\sigma}}{dy}(\nu q \to \mu q'). \tag{6.123}$$

Therefore,

$$\frac{d\sigma}{dx\,dy}(\nu p \to \mu^- X) = \frac{G^2 xs}{\pi}\,[d(x) + (1-y)^2\,\bar{u}(x)], \tag{6.124}$$

$$\frac{d\sigma}{dx\,dy}(\bar{\nu} p \to \mu^+ X) = \frac{G^2 xs}{\pi}\,[\bar{d}(x) + (1-y)^2\,u(x)]. \tag{6.125}$$

For neutron target, u and d are interchanged. The weak cross-sections for ν and $\bar{\nu}$ scattering from isoscalar target can be written in the form

$$\frac{d^2(\sigma^\nu + \sigma^{\bar{\nu}})}{dx\,dy} = \frac{G_F^2 s}{\pi}\,[(1-y)^2 F_2^\nu + xy^2 F_1^\nu], \tag{6.126}$$

$$\frac{d^2(\sigma^\nu + \sigma^{\bar{\nu}})}{dx\,dy} = \frac{G_F^2 s}{2\pi}\,x F_3^\nu\,(2y - y^2). \tag{6.127}$$

Assuming that we are below charm threshold and setting the Cabibbo angle to zero, the neutrino relations are

$$\left.\begin{aligned} F_2^{\nu p}(x) &= F_2^{\bar{\nu}n}(x) = 2x[d(x) + \bar{u}(x)], \\ F_2^{\nu n}(x) &= F_2^{\bar{\nu}p}(x) = 2x[u(x) + \bar{d}(x)], \\ F_3^{\nu p}(x) &= F_3^{\bar{\nu}n}(x) = 2[d(x) - \bar{u}(x)], \\ F_3^{\nu n}(x) &= F_3^{\bar{\nu}p}(x) = 2[u(x) - \bar{d}(x)]. \end{aligned}\right\} \tag{6.128}$$

For comparison, we give below the isoscalar structure function for the case of the electron Deep Inelastic Scattering (DIS),

$$\begin{aligned} F_2^{eN} &= \frac{1}{2}[F_2^{ep} + F_2^{en}] = \frac{x}{9}[5(u(x) + \bar{u}(x) + d(x) + \bar{d}(x)) + 2(s(x) + \bar{s}(x))] \\ &\approx \frac{5}{18}x[q(x) + \bar{q}(x)] \end{aligned} \tag{6.129}$$

neglecting strange quark contribution. Using SLAC-MIT results, we find that $\int s\,dx\,F_2^{eN}$ $= 0.15 \pm 0.01$, whereas the Gargamelle Experiments at CERN yield $\int dx\,F_2^{\nu N}(x) = 0.49 \pm 0.05$. The ratio is 3.3 ± 0.4 and is consistent with 18/5 = 3.6 expected from the parton model prediction. Note that the integral of F_2 in the neutrino scattering directly gives the momentum fraction carried by the u and d quarks which is approximately 0.54 compatible with the above result and justifying the assumption that the strange quark momentum fraction is indeed negligible.

For lepton scattering from isoscalar targets, therefore, we get

$$\frac{F_2^e}{F_2^\nu} \equiv \frac{(F_2^{ep} + F_2^{en})}{(F_2^{\nu p} + F_2^{\nu n})} = \frac{5}{18}\left[\frac{(q + \bar{q}) + (2/5)(s + \bar{s})}{q + \bar{q}}\right], \tag{6.130}$$

where the factor 5/18 is just the average of the squared charges for the u and d quarks and

$$q \equiv q(x) = u(x) + d(x), \tag{6.131}$$

$$\bar{q} \equiv \bar{q}(x) = \bar{u}(x) + \bar{d}(x). \tag{6.132}$$

Thus if the strange quarks can be ignored,

$$F_2^e = 5/18 F_2^\nu. \tag{6.133}$$

There is strong experimental support for this relation when $x \geq 0.1$, indicating that the strange quark content of nucleons is small, except possibly near $x = 0$, where they may be produced by sea quark excitation. The structure function $F_3^{\nu N}$ is interesting as it provides yet another independent measure of quark densities. Since it is the difference of quark and antiquark densities, we can rewrite F_3 as

$$F_3^{\nu N} = q(x) - \bar{q}(x) = u_v(x) + d_v(x). \tag{6.134}$$

The integral of F_3 simply gives the number of valence quarks. Hence,

$$\int dx F_3^{\nu N}(x) = 3 \text{ (the number of valence quarks)}$$
$$= 3.2 \pm 0.5 \text{ (from the CDHS data).} \tag{6.135}$$

The sum rule is known as the Gross-Llewellyn-Smith sum rule. The neutral-current processes may also be studied similarly.

6.3.5 e^+e^- Physics

In the deep inelastic scattering of leptons by hadrons, the large q^2 short wavelength virtual photon prepared by the inelastic scattering probes the hadron, revealing its constituents. High resolution photons can also be prepared by colliding high energy electron and positron beams head on. The e^+e^- colliders can be used to study QED, weak interactions, quarks and gluons and also to study or search for heavy quarks and leptons. Moreover, e^-e^+ annihilation is a clean process in the sense that leptons rather than hadrons which are complex structure made of partons, appear in the initial state.

$e^-e^+ \to$ *any hadron*

We would expect that this process to be described by Fig. 6.6, where the quarks subsequently produce hadrons in the final state. If we ignore the QCD corrections, we can obtain the cross-section formulae for $e^+e^- \to q\bar{q}$ by comparing it with electron scattering by muon with which we are already familiar. Then the differential cross-section is

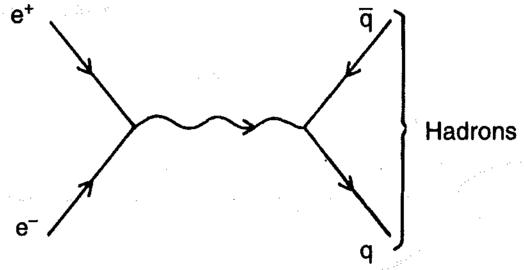


Fig. 6.6 Electron-positron annihilation into hadrons.

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|M_{fi}|^2 = \frac{\alpha^2}{4s}(1+\cos^2\theta), \tag{6.136}$$

in the centre of mass system. The total cross-section is then obtained by integrating over the CM scattering angle θ:

$$\sigma_{\text{tot}} = \frac{4\pi\alpha^2}{3s}. \tag{6.137}$$

We can now go to the $e^+e^- \to q\bar{q}$ subprocess. The difference is the charge factor. At high energies the quark-antiquark pairs, $u\bar{u}, d\bar{d}, s\bar{s}, c\bar{c}, b\bar{b}$ are produced. Thus the inclusive $e^+e^- \to$ hadrons, where all possible final states are summed over, are given by

$$\sigma_{e^+e^- \to \text{hadrons}} = \sigma_{e^+e^- \to \mu^+\mu^-} \Sigma_q e_q^2 = \frac{4\pi\alpha^2}{3s} \Sigma_q e_q^2 . \tag{6.138}$$

6.3.6 Production of Lepton Pairs

The Drell-Yan process

One of the most important reactions actively studied in a number of experiments is

$$p\bar{p} \to \mu^+\mu^- \; (e^+e^-) + \text{hadrons}. \tag{6.139}$$

To lowest order in the electronic charge e, this occurs through the production of a virtual photon.

$$p\bar{p} \to \gamma + \text{hadron}. \tag{6.140}$$

The photon subsequently produces lepton pairs $\mu^+\mu^-$ or e^+e^-. In the parton model, the virtual photon is produced by the annihilation of a quark from one proton with the antiquark from the other proton. This is the Drell-Yan mechanism (Fig. 6.7). We can represent the cross-section in terms of point-like scattering

$$\sigma(q\bar{q} \to l^+l^-) = \frac{4\pi\alpha^2}{3q^2} e_i^2 \tag{6.141}$$

where $e_i, - e_i$ are the electric charges of the $q\bar{q}$ pair. The differential cross-section is then

$$\frac{d\sigma}{dq^2}(q\bar{q} \to l^+l^-) = \frac{4\pi\alpha^2}{3q^4} e_i^2 . \tag{6.142}$$

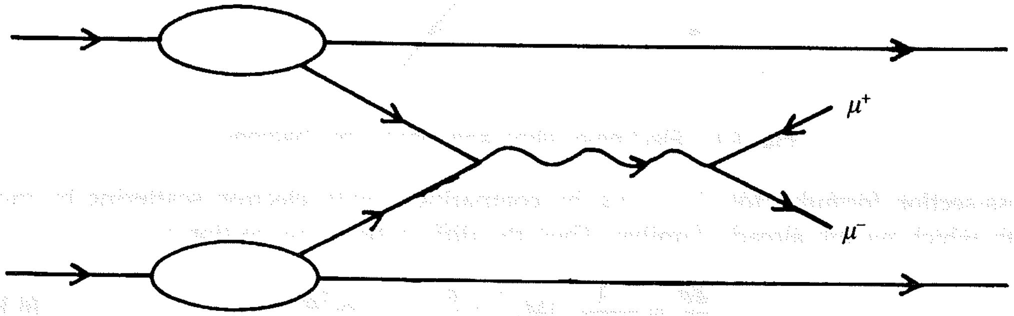


Fig. 6.7 Drell-Yan process.

The value of the mass squared of the virtual photon, $-q^2$, is related to the proton momenta and the momentum fractions x_i, $\bar{x}_i$ carried by the constituents. If P_1, P_2 are the four-momenta of the initial protons, then the four momenta of the constituent quark-antiquark are $k_i = x_i P_1$ and $\bar{k}_i = \bar{x}_i P_2$ so that the invariant mass squared of the photon and hence of the dilepton is

$$m^2 = - q^2 = - (k_i + k_i)^2 = - (x_i P_1 + \bar{x}_i P_2)^2 = - (2P_1 P_2 x_i \bar{x}_i + P_1 x_i^2 + \bar{x}_i P_2) \tag{6.143}$$

The total centre of mass energy squared of the proton-proton system is

$$s = -(P_1 + P_2)^2 = -(2P_1P_2 + P_1^2 + P_2^2). \tag{6.144}$$

If $s >> M^2$, $m^2 >> x^2M^2$, then we get

$$m^2 = x_i\bar{x}_i s. \tag{6.145}$$

We obtain for the lepton-pair cross-section in the $p\bar{p}$ collision, the relation

$$\frac{d\sigma}{dq^2} = \sum_i e_i^2 \int_0^1 dx_i \int_0^1 d\bar{x}_i f_i^1 f_i^2 \frac{4\pi\alpha^2}{3m^2}\,\delta(m^2 - x_i\bar{x}_i s). \tag{6.146}$$

We have two levels of prediction:

1. The cross-section is predicted to scale as

$$m^4 \frac{d\sigma}{dm^2}\left(\text{or } m^3 \frac{d\sigma}{dm}\right) \sim f(m^2/s). \tag{6.147}$$

That is, it depends on the dimensionless parameter.

2. Since the quark distribution functions can be obtained from the analysis of the inelastic lepton scattering data, the absolute size and energy dependence of $(d\sigma/dm^2)$ $(p\bar{p} \to \mu^+\mu^- \ldots)$ can be predicted. Upon comparing the data, it is found that the cross-section determines an accurate determination of antiquark distribution.

The assumption that the lepton pair production proceeds via quark-antiquark annihilation to a virtual photon is verified in these experiments by observing the polar angular distribution of either lepton in the dilepton rest frame, relative to the incident proton beam direction. As in the inverse process $e^+e^- \to q\bar{q}$, via single photon exchange, we expect an angular distribution of the form

$$\frac{d\sigma}{d\Omega} = \text{const. } (1 + \cos^2\theta), \tag{6.148}$$

as is indeed observed. The Drell-Yan process is of supreme importance. One reason for this is that the antiquark distribution enters multiplicatively so that cross-section is sensitive to them. The measurement of differential cross-section gives a more accurate determination of the antiquark distribution than is possible for lepton scattering where the antiquark enters additively rather than multiplicatively.

6.3.7 The Quark Parton Model and Hadron Production

We have considered the fully inclusive processes where the final hadronic state was summed over. Interesting possibilities arise when we consider the specific hadron final states. Consider, as an example, $e^+e^- \to hX$, where h is a hadron and X the debris in which we are not interested. The relevant diagrams are shown in Fig. 6.8.

The Quark Parton Model (QPM) can also describe hadron production through the fragmentation of a quark and/or antiquark into hadrons. For the fragmentation of quarks

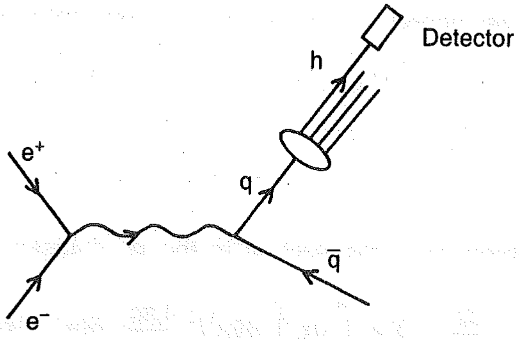


Fig. 6.8 Inclusive hadron production in e^-e^+ annihilation.

into hadrons, we use the analogous formalism to that used earlier to describe quarks inside hadrons. If only a certain hadron (e.g. $\pi^\pm, K^\pm, K^0$) is looked for in the e^+-e^- annihilation, the cross-section is calculated from the cross-section for the pair production of quark and antiquark and the probability that a quark i will convert into a hadron h. We write,

$$\frac{d\sigma}{dz}(e^+e^- \to hX) = \sum_i \sigma(e^+e^- \to q_i\bar{q}_i)\,[D_i^h(z) + D_{\bar{i}}^h(z)], \tag{6.149}$$

Here, i stands for the quark flavour. The fragmentation function D represents the probability that the hadron h is produced by the quark (or antiquark) carrying a fraction z of its energy; $z = E_h/E_q$. It describes the transition, parton → hadron, in the same way as the structure function $f(x)$ does for the transition hadron → quark. Like the f, D's are also subject to the constraints

$$\sum_i \int_{z_{\text{min}}}^{1} dz\,[D_i^h(z) + D_{\bar{i}}^h(z)] = n_h n, \tag{6.150}$$

$$\sum_i \int_{z_{\text{min}}}^{1} dz\,z\,D_i^h(z) = 1 \tag{6.151}$$

imposed by the probability and momentum conservation. z_{min} is related to the threshold energy $2m_h/Q$ for the production of a hadron of mass m_h.

$$\frac{d\sigma}{dz}(e^+e^- \to hX) = \sum_i e_i^2\sigma(e^+e^- \to \mu^+\mu^-)\,[D_i^h(z) + D_{\bar{i}}^h(z)], \tag{6.152}$$

$$\frac{1}{\sigma}\frac{d\sigma}{dz}(e^+e^- \to hX) = \sum_i e_i^2[D_i^h(z) + D_{\bar{i}}^h(z)]/\sum_i e_i^2 = f(z). \tag{6.153}$$

Now, if there are only u and d quarks and pions,

$$\frac{d\sigma}{dz}(e^+e^- \to \pi) = \sigma(e^+e^- \to \mu^+\mu^-)[e_u^2\{D_u^{\pi^+} + D_{\bar{u}}^{\pi^+} + D_u^{\pi^-} + D_{\bar{u}}^{\pi^-}\} + e_d^2\{D_d^{\pi^+} + D_{\bar{d}}^{\pi^+} + D_d^{\pi^-} + D_{\bar{d}}^{\pi^-}\}]. \tag{6.154}$$

Isospin invariance gives

$$D_u^{\pi^+} = D_{\bar{u}}^{\pi^-} = D_d^{\pi^-} = D_{\bar{d}}^{\pi^+}, \tag{6.155}$$

$$D_u^{\pi^-} = D_d^{\pi^+} = D_{\bar{u}}^{\pi^+} = D_{\bar{d}}^{\pi^-} \tag{6.156}$$

so that

$$\frac{d\sigma}{dz}(e^+e^- \to \pi) = \sigma(e^+e^- \to \mu^+\mu^-)(e_u^2 + e_d^2)2[D_u^{\pi^+} + D_u^{\pi^-}], \tag{6.157}$$

$$\frac{d\sigma}{dz}(e^+e^- \to \pi) = \sigma(e^+e^- \to \mu^+\mu^-)2(e_u^2 + e_d^2)D_u^{\pi}, \tag{6.158}$$

where

$$[D_u^{\pi^+} + D_u^{\pi^-}] = D_u^{\pi}. \tag{6.159}$$

Similar derivation yields, *for neutrino scattering*, the relation

$$\frac{1}{N_\nu}\frac{dN}{dz} = D_u^{\pi}, \tag{6.160}$$

for antineutrino scattering, the equation

$$\frac{1}{N_{\bar{\nu}}}\frac{dN}{dz} = D_u^{\pi}, \tag{6.161}$$

for electron or muon scattering, the relation

$$\frac{1}{N}\frac{dN}{dz} = D_u^{\pi}. \tag{6.162}$$

These simple results allow a direct test of the basic QPM conjecture. Data from different experiments show good agreement. The fragmentation functions $D(z)$ describing the properties of partons are the same no matter how the partons are produced. Consider the inclusive lepto-production. We can see that lepto-production cross-section (see Fig. 6.9) is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dz}(e^-p \to hX) = [\sum_i e_i^2 f_i(x)\, D_i^h(z)]/[\sum_i e_i^2 f_i(x)]. \tag{6.163}$$

Here, $f_i(x)$ are the proton structure functions. The sum runs over the quarks and antiquarks that can be a parent of h.

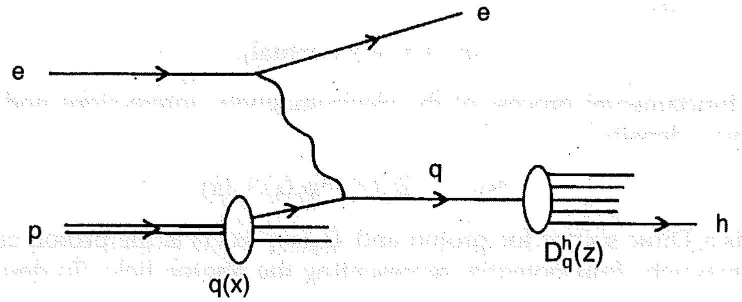


Fig. 6.9 Lepto-production.

Weak hadronic currents

We shall now see that the observed semi-leptonic weak interactions are well described to lowest order by an effective interaction Hamiltonian which couples the lepton current to a weak hadron current, which contains both vector and axial vector parts. To do that, let us first discuss simpler aspects of weak interactions.

6.4 The Weak Interactions

Though the first weak process was observed by H. Becquerel in 1896, the starting point of the weak interaction theory was the suggestion in 1933 by W. Pauli that a new neutral particle with spin 1/2 and zero rest mass was emitted together with electron in the β-decay of nuclei. With the introduction of this new particle (called neutrino—a little neutron—by E. Fermi), one could explain the observation of the continuous energy spectrum of the electron as also ensure the conservation of both the energy and the angular momentum. In the same year, Fermi formulated a quantum field theory of β-decay in close analogy with quantum electrodynamics. The Fermi theory remains with some minor changes a fundamental element of our theoretical knowledge of weak processes. We shall denote by ν_e and $\bar{\nu}_e$ neutrino and antineutrino emitted with a positron e^+ and an electron e^-, respectively in the reactions:

$$(A, Z) \rightarrow (A, Z-1) + e^+ + \nu_e,$$

$$(A, Z) \rightarrow (A, Z+1) + e^- + \bar{\nu}_e,$$

where (A, Z) represents a nucleus with atomic number A and electric charge Z.

6.4.1 Fermi's Theory of β-radioactivity

The elementary process of β-radioactivity is the reaction

$$n \rightarrow p + e^- + \bar{\nu}_e \quad \text{(electron emission)}$$

or

$$p \rightarrow n + e^+ + \nu_e \quad \text{(positron emission)}$$

The emission of light particles e and $\bar{\nu}_e$ in nuclear β-decay was described by Fermi in a way analogous to the emission of a photon. The process

$$n \rightarrow p + e^- + \bar{\nu}_e$$

is taken to be similar to

$$p \rightarrow p + \gamma \text{ (virtual).}$$

which is the fundamental process of the electromagnetic interactions and is described by the Hamiltonian density

$$\mathcal{H}_{\text{EM}} = e\bar{\psi}_p(x)\gamma^\mu\psi_p(x)A_\mu(x),$$

where $\psi_p(x)$ is a Dirac spinor for proton and $\bar{\psi}_p(x)\gamma^\mu\psi_p(x)$ is the proton current and $A_\mu(x)$ is the electromagnetic four potential representing the photon field. To describe the process

$n \to p + e^- + \bar{\nu}_e$, Fermi replaced $A_\mu(x)$ by a four vector $\bar{\psi}_e(x)\gamma_\mu\psi_\nu(x)$ built from the electron and neutrino fields. He wrote, for the Hamiltonian density, the equation

$$\mathcal{H}(x) = G_F(\bar{\psi}_p(x)\gamma^\mu\psi_n)\,(\bar{\psi}_e(x)\gamma_\mu\psi_\nu(x)), \tag{6.164}$$

where G_F is the weak interaction coupling constant and is known as the Fermi constant. (G_F has the dimension of inverse mass squared.) This interaction was called the four-fermion interaction because four spin 1/2 fields are interacting at the single space time point. However, it became soon clear that the coupling equation (6.164) was not adequate to reproduce the various β-decay processes, and Fermi proposed the more general (but still parity conserving) Hamiltonian density

$$\mathcal{H}(x) = G_F\sum_i C_i(\bar{\psi}_p(x)\Gamma^i\psi_n(x))(\bar{\psi}_e(x)\Gamma_i\psi_\nu(x)), \tag{6.165}$$

where $C_1 = C_S$, $C_2 = C_V$, $C_3 = C_A$, $C_4 = C_T$, $C_5 = C_P$. All C_i are real if time reversal invariance is assumed. Γ_i are the 16 independent 4 × 4 matrices. The Dirac bilinears $\bar{\psi}(x)\Gamma_i\psi(x)$ transform under the Lorentz group (including space inversion), as indicated in Table 6.2.

Table 6.2 Dirac Bilinears

Γ_i	*Dirac bilinears* ($\bar{\psi}\Gamma_i\psi$)	*Tensor*	*No. of components*
1	$\bar{\psi}(x)\psi(x)$	Scalar	1
γ_μ	$\bar{\psi}(x)\gamma_\mu\psi(x)$	Four vector	4
$\gamma_\mu\gamma_5$	$\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$	Axial vector	4
$\sigma_{\mu\nu}$	$\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)$	Antisymmetric tensor	6
γ_5	$\bar{\psi}(x)\gamma_5\psi(x)$	Pseudoscalar	1

If we neglect Coulomb interaction of the electron, the outgoing leptons can be described by the plane wave

$$\psi_l(x) = \frac{1}{\sqrt{V}}e^{ip\cdot x}u(\mathbf{p}). \tag{6.166}$$

Spinor $u(p)$ is a solution of the Dirac equation. The nucleon fields, on the other hand, vanish outside the nuclear volume V_N. The Hamiltonian then can be rewritten as

$$H_W = G_F\sum_i C_i\bar{u}_l(p)\Gamma^i u(q)\int_{V_N} d^3x e^{i(p+q)\cdot x}\bar{\psi}_p(x)\Gamma_i\psi_n(x)). \tag{6.167}$$

The energies involved in the β-decay are such that the nucleon can be treated non-relativistically and the magnitude of **p** and **q** are of the order of electron Compton wavelength $\lambda_e = 3.8 \times 10^{-11}$ cm. For a nucleus with radius of the order of 10×10^{-13} cm, the exponent in Eq. 6.166 is $(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x} = (R_N/\lambda) = 2.5 \times 10^{-2}$, so that we can replace the exponent by 1. To summarize, we make the following approximations:

1. The nucleon spinors are treated in the non-relativistic limit.
2. The exponent corresponding to the lepton fields is replaced by 1.

This then corresponds to the allowed transitions. It may so happen that the contribution in such an approximation is zero. Then one takes into account higher terms in the series expansion of the exponential. One is then dealing with forbidden transitions. For the allowed transitions, the matrix element

$$M_{fi} = \langle f | H_W | i \rangle \tag{6.168}$$

is independent of the momenta **p** and **q**. The transition probability is given by

$$\omega_{fi} = \int d^3p \int d^3q \delta(E_0 - E_e - E) | \langle f | H_W | i \rangle |^2 (E_0 - E_e)^2 \tag{6.169}$$

where integration is over all variables except the electron energy and E_0 is the maximum value for E_e (neglecting nuclear recoil). Therefore, the energy spectrum of electron is statistical and the quantity

$$\frac{1}{p}\left(\frac{d\omega}{dp}\right)^{1/2} \sim (E_0 - E_e) \tag{6.170}$$

is represented by a straight line with respect to E_e. This is called the Kurie plot and is in agreement with experiment.

In order to incorporate parity violation in weak processes, we need to add a pseudoscalar term to the Hamiltonian density in (6.165) so that it does not have a well-defined parity. The Hamiltonian density can then be written as

$$\mathcal{H}(x) = G_F \sum_i C_i (\overline{\psi}_p(x) \Gamma^i \psi_n)(\overline{\psi}_e(x) \Gamma_i \psi_\nu(x)) + G_F \sum_i C_i' (\overline{\psi}_p(x) \Gamma^i \gamma_5 \psi_n)(\overline{\psi}_e(x) \Gamma_i \psi_\nu(x)). \tag{6.171}$$

6.4.2 Classification of Weak Processes

Weak interactions group themselves phenomenologically into three categories:

(i) Purely leptonic processes

Here only leptons are involved. The best known example is the muon decay

$$\mu^- \to e^- + \overline{\nu}_e + \nu_\mu,$$

$$\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu.$$

Leptonic decays are induced by self-interaction of the leptonic current

$$L_\mu = \bar{e}\gamma_\mu(1 - \gamma_5)\nu_e + \bar{\mu}\gamma_\mu(1 - \gamma_5)\nu_\mu + \bar{\tau}\gamma_\mu(1 - \gamma_5)\nu_\tau.$$

(ii) Semi-leptonic processes

These processes involve hadrons and a lepton pair like $e\overline{\nu}_e$. They are further classified according to the change of flavour. For example,

(a) $\Delta S = 0$. *Strangeness conserving.* These transitions include β-nuclear decays. The elementary process is

$$n \to p + e^- + \bar{\nu}_e$$

- Electron capture: $e^- + p \to n + \nu_e$.
- μ-capture: $\mu^- + p \to n + \nu_\mu$.
- Neutrino reactions:
- Elastic processes: $\nu_\mu + n \to p + \mu^-$, $\bar{\nu}_\mu + p \to n + \mu^+$.
- Inelastic processes: $\pi \to \mu\nu$ or $e\nu$, $\Sigma^- \to \Lambda + e^- + \bar{\nu}_e$.

(b) *Flavour changing processes.*

Strangeness changing $|\Delta S| = 1$

$$\Lambda^0 \to p + e^- + \bar{\nu}_e, \qquad K^+ \to \mu^+ + \nu_\mu, \qquad K^+ \to \pi^0 + \mu^+ + \nu_\mu.$$

Charm changing $|\Delta c| = 1$, $|\Delta S| = 0$

$$\Lambda_c^+ \to \Lambda^0 + e^+ + \nu_e, \qquad D^+ \to \mu^+ + \nu_\nu.$$

Beauty changing $|\Delta b| = 1$, $|\Delta S| = 1$

$$\Lambda_b \to \Lambda_c^+ + e^- + \bar{\nu}_e.$$

(iii) Non-leptonic weak processes

$|\Delta S| = 1$

$$\Lambda^0 \to p + \pi^-, \qquad K^- \to \pi^- + \pi^0, \qquad K_s^0 \to \pi^+ + \pi^-;$$

$|\Delta C| = 1$

$$D^0 \to K^- + \pi^+, \qquad \Lambda_c^+ \to \Lambda + \pi^+, \qquad \Lambda_c^+ \to \Lambda + K^+ + \pi^0;$$

$|\Delta b| = 1$

$$\Lambda_b \to \Lambda_c^+ + \pi^-.$$

6.4.3 Two-Component Theory of Neutrino

H. Weyl hypothesized that only one of the two two-component equations is actually realized for a particle of zero mass, namely neutrino. It was rejected by W. Pauli in 1933 on the grounds that the equation of motion is not invariant under parity. The hypothesis was revived in 1957 by L.D. Landau, T.D. Lee and C.N. Yang and A. Salam. The neutrino field would appear in the form

$$\frac{1}{2}(1 - \gamma_5)\psi$$

if it is left handed, or in the form

$$\frac{1}{2}(1 + \gamma_5)\psi$$

if it is right handed. The actual form is to be decided by experiment.

6.4.4 Determination of Helicity of Neutrino

We will now consider the experimental evidence that neutrino is left handed, i.e. it has helicity $h = -1$. The experiment was performed by Goldhaber, Grodyzin, and Sunyar[1] in 1958. Consider the reaction

$$e^- + p \rightarrow n + \nu_e.$$

This is what happens when a K-shell electron is captured by a nucleus A.

$$e^- + A \rightarrow B^* + \nu_e,$$

$$J^A = 0, \qquad J^{B^*} = 1,$$

leaving an excited nucleus B^*. Suppose these nuclei have spins 0 and 1 as indicated. Then B^* will decay to its spin zero ground state with the emission of a photon.

$$B^* \rightarrow B + \gamma.$$

The overall reaction is

$$e^- + A \rightarrow B + \nu_e + \gamma$$

which takes place through a resonant channel B^*. The resonant condition will be fulfilled when photon and neutrino move away in the opposite directions. The principle of the experiment is to replace the detection of a recoil nucleus by the measurement of circular polarization of a γ emitted by the recoiling nucleus. We use the conservation of J_z to show that both the photon and neutrino in the reaction have the same helicity. Photon has spin 1. But because it is massless, it has for the third component of spin only values ±1 (transverse polarization). Choose the direction of neutrino emission as the z-axis. If the neutrino is left handed, it has $J_z = -1/2$ and the conservation of J_z demands

$$e^- + A \rightarrow B + \nu_e + \gamma$$

$$J_z = 1/2 \quad 0 \quad -1/2 \quad +1$$

So γ has $S_z = 1$ but is moving along the negative z-axis, and so has negative helicity as in the case of neutrino. On the other hand, if the neutrino is right handed, we must have

$$e^- + A \rightarrow B + \nu_e + \gamma$$

$$J_z = -1/2 \quad 0 \quad +1/2 \quad -1$$

The photon now has positive helicity again like the neutrino. In either case then, measuring the photon helicity amounts to measuring the neutrino helicity. Goldhaber, et al. found nuclei with required spins. E^{52} has $J^P = 0^-$ and gives an excited state Sm^{152*} on K-capture. This then decays to the ground state with $J^P = 0^+$. The emitted photon was found to have negative helicity. So it follows that the neutrino has negative helicity, i.e. the neutrino is left handed.

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[1] M. Goldhaber, L. Grodyzin and A. Sunyar (1958): *Phys. Rev.*, **109**, 1015.

6.5 Form of Four-Fermion Weak Coupling

It is necessary that the fermion field appear only in the form $1/2\,(1-\gamma_5)\psi$. Further, because $[(1-\gamma_5)\psi]^\dagger = \bar{\psi}(1+\gamma_5)$, $\bar{\psi}$ must appear only in the form $1/2\ \bar{\psi}(1+\gamma_5)$. Since $(1+\gamma_5)(1-\gamma_5) = 0$, the only non-vanishing bilinear form constructed out of two different spinors is

$$\bar{\psi}_1\gamma_\mu(1-\gamma_5)\psi_2.$$

All other bilinear forms vanish because

$$\begin{aligned}(1+\gamma_5)1(1-\gamma_5) &= 0,\\ (1+\gamma_5)\gamma_5\,(1-\gamma_5) &= 0,\\ (1+\gamma_5)\sigma_{\mu\nu}(1-\gamma_5) &= 0.\end{aligned}$$

This means that the four-fermion interaction is of the form

$$\bar{\psi}_1\gamma_\mu(1-\gamma_5)\psi_2\bar{\psi}_3\gamma_\mu(1-\gamma_5)\psi_4,$$

as was stated in 1958 by E.C.G. Sudarshan and R.E. Marshak, R.P. Feynman and M. Gell-Mann; and J.J. Sakurai before the confirming experiments were performed.

The above interaction is a point interaction and may be taken as a phenomenological effective interaction valid only in the low momentum transfer limit. If the interaction between the two currents J_{12} and J_{34} is assumed to be transmitted via the exchange of a vector boson W and if the boson is massive enough, then the effective interaction would appear to be a point interaction.

The problem

If

$$\mathcal{H}_{\text{int}} = \bar{\nu}\gamma_\mu(1-\gamma_5)\mu\bar{e}\gamma_\mu(1-\gamma_5)\nu_e. \tag{6.172}$$

Show that this equation can also be written as

$$\mathcal{H}_{\text{int}} = \bar{\nu}\gamma_\mu(1-\gamma_5)\nu\bar{e}\gamma_\mu(1-\gamma_5)\mu.$$

(The V-A coupling is permutation symmetric.)

6.5.1 Structure of Nucleon Weak Current

Within a year after the discovery of parity violation, a large amount of experimental information had been accumulated showing that V and A couplings were responsible for nuclear β-decay. The matrix element for the general parity violating weak Hamiltonian then reduces to

$$\left.\begin{aligned}M &= \frac{G}{\sqrt{2}}\,[C_V\bar{u}_p\gamma_\mu u_n\bar{u}_e\gamma^\mu(1-\gamma_5)u_\nu + C_A\bar{u}_p i\gamma_\mu\gamma_5 u_n\bar{u}_e i\gamma^\mu(1-\gamma_5)u_\nu]\\ &= \frac{G}{\sqrt{2}}\,[\bar{u}_p\gamma_\mu(C_V - C_A\gamma_5)u_n\bar{u}_e\gamma^\mu(1-\gamma_5)u_\nu.\end{aligned}\right\} \tag{6.173}$$

C_V and C_A are normalized vector and axial vector coupling constants. $C_V = 1$ since the value of G is deduced from the pure Fermi transition

$$O_8^{14} \rightarrow N_7^{14} + e^+ + \nu$$

in good agreement with the value deduced from μ-lifetime. The ratio of the absolute values of C_A and C_V determines the relative strength of Gamow-Teller and Fermi couplings. It was determined by comparing the half-lives or, more exactly, the *ft*-values of O^{14}, which is a pure Fermi emitter, and C^{12} which is a mixture of Fermi and Gamow-Teller transitions. We find that

$$|C_A/C_V| = 1.2.$$

The sign of C_A/C_V is related to the longitudinal polarization of the proton in the neutron β-decay. However, the velocity of the proton is too small; $v/c \sim 0.04$ to allow such a measurement. Experiments on the decay of polarized neutrons were performed in 1958 at Argonne and Chalk River Laboratories. The observed angular distribution with respect to neutron spin direction has a very small asymmetry $A \sim -0.11$ in

$$1 + A\frac{\mathbf{n}\cdot\mathbf{v}_e}{c},$$

where $\mathbf{n}$ is the unit vector representing the spin direction of the decaying nucleus and $\mathbf{v}_e$ is the velocity of the emitted electron. There is destructive interference between Fermi and Gamow-Teller terms here. From this effect, one concludes that the ratio C_A/C_V is negative. The best value of C_A/C_V now is

$$C_A/C_V = -1.250 \pm 0.009$$

6.5.2 *V-A* Law

The two-component theory of neutrino may be a consequence of a universal Fermi interaction between pairs of fermions with equal amounts of vector and axial vector couplings. This *V-A* law was postulated in 1957–58 even before the experimental situation had been settled. $C_A = -1.25$ in nuclear β-decay was thought to be a 'renormalization effect' due to the strong interactions. In the absence of strong interactions, one would have $C_A = -C_V$, and hence *V-A*. The almost perfect agreement between the value of Fermi constant deduced from μ-decay and O^{14}–β decay ($C_V = 1$) shows that there is no renormalization effect on the vector part of the current. This was interpreted by the conserved vector current hypothesis of Feynman and Gell-Mann (1958).

6.6 Current Form of Weak Interactions

The μ-decay is described by the product of two leptonic currents

$$\bar{\mu}\gamma_\mu(1 - \gamma_5)\nu_\mu \text{ and } \bar{e}\gamma_\mu(1 - \gamma_5)\nu_e.$$

The neutron decay is described by the product of two weak currents, the nucleon current which is nearly

$$\bar{p}\,\gamma_\mu\,(1-\gamma_5)n$$

and the lepton current

$$\bar{e}\,\gamma_\mu\,(1-\gamma_5)\nu_e.$$

6.6.1 Universal Current-Current Weak Interaction

Each current is a sum of vector and axial vector parts. Feynman and Gell-Mann, Sudarshan, and Marshak generalized this as current-current hypothesis. The weak current is a sum of two parts: a leptonic-current

$$l_\mu = \bar{e}\gamma_\mu(1-\gamma_5)\nu_e + \bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu$$

and a hadronic-current $h_\mu = \bar{p}\,\gamma_\mu(1-\gamma_5)n$ + terms corresponding to strange particle decays and other flavour decays. In terms of quarks, we may write the hadronic current as

$$h_\mu = \bar{u}\gamma_\mu(1-\gamma_5)d + \bar{u}\gamma_\mu(1-\gamma_5)s.$$

The purely leptonic *muon decay*

$$\mu(p) \to e^-(p') + \bar{\nu}_e(k') + \nu_\mu(k)$$

is well described, to lowest order, by an interaction Hamiltonian which couples a current $a(l_\mu)$ with another current (l^μ). The factor $(1-\gamma_5)$ expresses the two-component nature of neutrinos and gives to each current a vector and axial vector part. The Hamiltonian density for the μ-decay is thus

$$\mathcal{H}_W = \frac{G}{\sqrt{2}}\,[\bar{\psi}_{\nu_\mu}(x)\gamma^\mu(1-\gamma^5)\psi_\mu(x)][\bar{\psi}_e(x)\gamma_\mu(1-\gamma^5)\psi_{\nu_e}(x)]. \tag{6.174}$$

The matrix element in the momentum space is given by

$$M = \frac{G}{\sqrt{2}}\,[\bar{u}(k)\gamma^\mu(1-\gamma^5)u(p)][\bar{u}(p')\gamma_\mu(1-\gamma^5)v(k')]. \tag{6.175}$$

From the above expression of the matrix element we can calculate the decay by the standard calculations; some details of these are given in Appendix E. The decay rate then turns out to be

$$\Gamma^{(0)} \equiv \frac{1}{\tau_\mu} = \frac{G^2 m_\mu^5}{192\pi^3}.$$

Radiative corrections to the muon decay are important,[1] giving for the decay rate the corrected expression

$$\Gamma = \Gamma^{(0)}\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right) + \frac{3}{5}\frac{m_\mu^2}{m_W^2}\right], \tag{6.176}$$

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[1] Scheck, F. (1978): Muon Physics, *Physics Reports,* **44,** 187.

where $\Gamma^{(0)}$ is the uncorrected expression and the third term on the right-hand side is too small and can be ignored. In the general case, and including the elecron mass terms, the decay rate is given by

$$\Gamma = \frac{m_\mu^5 G_F^2}{192\pi^3}\frac{A}{16}\left[1 + 4\frac{m_e}{m_\mu}\frac{\alpha - 2\beta}{A} - \frac{8\,m^2}{m_\mu^2} + O\left(\frac{m_e^3}{m_\mu^3}\right)\right]. \tag{6.177}$$

For V-A interaction, $A = 16$, $\alpha = \beta = 0$. With $\tau_\mu = 2.19714 \times 10^{-6}$ s, we can extract G from this datum, viz.

$$G_F \cong 10^{-5}/M_N^2,$$

$$G_\mu = (1.16632 \pm .00002) \times 10^{-5}\,\mathrm{GeV}^{-2}.$$

The value of the coupling constant from the neutron β-decay turns out to be

$$G_\beta = (1.136 \pm .003) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

indicating the universality of the weak interaction coupling constant.

6.7 Strangeness Non-changing Decays: *V-A* law

Let us consider the decays

$$\pi \to \mu\nu_\mu, \qquad \pi \to e\nu_e.$$

According to the phenomenological current × current theory, we may expect

$$R = \frac{\Gamma(\pi \to e\bar{\nu})}{\Gamma(\pi \to \mu\bar{\nu})} \sim 1. \tag{6.178}$$

But, experimentally,

$$R = \frac{\Gamma(\pi \to e\bar{\nu})}{\Gamma(\pi \to \mu\bar{\nu})} = (1.247 \pm .028) \times 10^{-4}. \tag{6.179}$$

Qualitative explanation

The fermion wave function enters into the interaction in the form $(1 - \gamma_5)\psi$. This implies that in the relativistic limit of $v \to c$, all leptons are left handed and all antileptons are right handed. On the other hand, since pion is spinless, in the π-decay, $e^-(\mu^-)$ and $\bar{\nu}_e(\bar{\nu}_\mu)$ must have opposite helicity. Thus in the limit of $m_e \to 0$ and $m_\mu \to 0$, the decays are strictly forbidden. Since the mass of the electron is much closer to zero, we expect the decay rate for the electron mode to be very small whereas since the mass of the muon is not that small the muon mode could occur. We demonstrate in the following that this is indeed the case when we apply the V-A theory to the decay $\pi \to l\nu_l$. Since the helicity of $\bar{\nu}$ is +1 in the V-A theory, the emitted lepton in the π_{l2} decay, therefore, must have helicity, $h = +1$. However, in the limit of $v \to c$, a negative lepton must have $h = -1$ according to the V-A law. Now, because the electron emitted in the π_{l2} decay is quite relativistic, the electron decay will be strongly inhibited. On the other hand, muon emitted in the pion decay is not that relativistic and so may have the wrong helicity and, therefore, the muon decay is much less inhibited.

Quantitative explanation

Let us now consider the matrix element

$$S_{fi} = \frac{G}{\sqrt{2}} \int d^4x \langle l(q_l)\bar{\nu}_l(q_\nu)|\, j_\mu^{l+}(x)\, j^{\mu(h)}(x)|\pi^-(p)\rangle \tag{6.180}$$

for the decay $\pi \to l + \bar{\nu}$. For the integrand, we use the factorization

$$\langle l(q_l)\bar{\nu}_l(q_\nu)|\, j_\mu^{l+}(x)\, j^{\mu(h)}(x)|\pi^-(p)\rangle = \langle l(q_l)\bar{\nu}_l(q_\nu)|\, j_\mu^{l+}(x)|0\rangle \langle 0|j^{\mu(h)}(x)|\pi^-(p)\rangle \tag{6.181}$$

i.e., we write the matrix element as the product of the matrix element of the lepton current and that of the hadron current. Further

$$\begin{aligned}\langle 0\,|\,j^{\mu(h)}(x)\,|\,\pi^-(p)\rangle &= \langle 0\,|\,e^{iP\cdot x} j^{\mu(h)}(0) e^{-iP\cdot x}\,|\,\pi^-(p)\rangle = e^{-ip\cdot x}\,\langle 0\,|\,j^{\mu(h)}(0)\,|\,\pi^-(p)\rangle \\ &= e^{-ip\cdot x}\langle 0\,|\,A^{\mu(h)}(0)\,|\,\pi^-(p)\rangle = e^{-ip\cdot x}\; i f_\pi p^\mu.\end{aligned} \tag{6.182}$$

In the last but one step in this equation, the hadron current has been replaced by axial vector current because the pion is pseudoscalar, and so only the axial part of the current will contribute to the matrix element. For the last step we have used the Lorentz covariance. f_π is a Lorentz scalar and is a constant known as pion decay constant. Now,

$$\langle l(q_l)\bar{\nu}_l(q_\nu)|\, j_\mu^{l+}(x)\,|0\rangle = e^{i(q_l+q_\nu)\cdot x}\,\bar{u}_l(q_l)\gamma_\mu(1-\gamma_5)u_\nu(q_\nu).$$

From the conservation of four momenta, $p = q_l + q_\nu$. Substituting back into (6.181), we obtain for the matrix element (ME), the equation

$$\mathrm{ME} = \frac{G}{\sqrt{2}} f_\pi p^\mu \bar{u}_l(q)\gamma_\mu(1-\gamma_5)u_l(q_\nu) = \frac{G}{\sqrt{2}} f_\pi m_l \bar{u}_l(q)(1-\gamma_5)u_l(q_\nu).$$

Now $\gamma\cdot p = \gamma\cdot q_l + \gamma\cdot q_\nu$ and $\gamma\cdot q_l$ acting on spinor $\bar{u}_l(q)$ gives m_l, whereas $\gamma\cdot q_\nu$ gives zero. We, therefore, get

$$\sum_{\text{spin}} |\mathrm{ME}|^2 = G^2 f_\pi^2 m_l^2\, 4 q_l\cdot q_\nu \tag{6.183}$$

and the decay rate is given by

$$\Gamma = \frac{1}{8\pi} G^2 f_\pi^2 m_l^2 m_\pi \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2. \tag{6.184}$$

So,

$$R = \frac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.232 \times 10^{-4} \tag{6.185}$$

This ratio should be corrected due to radiative corrections. The corrected value turns out to be

$$R_{\text{corr.}} = 1.24 \times 10^{-4}, \tag{6.186}$$

in excellent agreement with the experimental value

$$R_{\text{expt.}} = (1.218 \pm 0.014) \times 10^{-4}. \tag{6.187}$$

f_π can be obtained from the μ mode. With $\tau_\pi = (2.604 \pm .007) \times 10^{-8}$, we find that $f_\pi = .93\, m_\pi = 90.8$ MeV. A very small value of the branching ratio R is a result of the vector and the axial vector nature of the weak current. It is easy to see that the scalar and pseudoscalar weak current gives a large value ~ 5 for this ratio. The small experimental value for R, therefore, is a strong evidence for the V-A theory of the weak interaction.

6.8 Strangeness Changing Decays

The strangeness changing decays, e.g.

$$\Sigma, \Lambda \to Ne\nu$$

$$\Xi \to \Lambda e\nu,$$

$$K \to l\nu,$$

$$K \to l\nu + n\pi$$

happen to occur at a slower rate as compared to the strangeness non-changing ones like

$$n \to p + e + \nu,$$

$$\Sigma \to \Lambda + e + \nu,$$

$$\pi \to \mu + \nu.$$

At the hadronic level, each semileptonic process is described by its corresponding weak hadronic current in terms of the hadron fields. At the quark level, however, we have a great simplification and all strangeness conserving semileptonic decays are governed by the current $J(\Delta S = 0) = u\bar{d}$, whereas all the strangeness changing ones are governed by the current $J(\Delta S = 1) = u\bar{s}$. We have suppressed the Lorentz structure like γ_μ and $\gamma_\mu\gamma_5$ in the quark currents. For the strangeness changing semileptonic weak transitions, the following qualitative features have been observed:

1. Their transition amplitudes have a strength of about a fifth of those for strangeness conserving decays;

2. They satisfy the selection rule $\Delta Q = \Delta S = \pm 1$.

(Change in the total charge of hadrons is equal to the change in total strangeness of hadrons.)

Since the baryon number is always conserved, from the Gell-Mann Nishijima relation

$$Q = I_3 + \frac{1}{2}(B + S)$$

and the selection rule (2), it follows that

$$\Delta I_3 = \Delta Q - \frac{1}{2}\Delta S = \pm\,\frac{1}{2}. \tag{6.188}$$

3. Actually, the stronger selection rule

$$|\Delta \mathrm{I}| = 1/2,$$

appears to be satisfied.

6.8.1 $K \rightarrow \mu\bar{\nu}_\mu$ decay

In analogy with the pion decay, we can write

$$\Gamma = \frac{1}{8\pi} G^2 f_K^2 m_l^2 m_K \left(1 - \frac{m_l^2}{m_K^2}\right)^2. \tag{6.189}$$

So,

$$R = \frac{\Gamma(K \rightarrow e\nu)}{\Gamma(K \rightarrow \mu\nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_K^2 - m_e^2}{m_K^2 - m_\mu^2}\right)^2 = 2.58 \times 10^{-5}. \tag{6.190}$$

This ratio should be corrected due to small radiative corrections and is again in excellent agreement with the experimental value

$$R_{\mathrm{expt}} = (2.425 \pm 0.012) \times 10^{-5}. \tag{6.191}$$

f_K can be obtained from the μ mode. With $\tau_K = (1.237 \pm .003) \times 10^{-8}$, and the branching fraction $\Gamma(K \rightarrow \mu\bar{\nu})/\Gamma(K \rightarrow \text{anything}) = 0.6350 \pm 0.0016$, we find that $f_K = 24.9$ MeV, giving

$$\frac{f_K}{f_\pi} = 0.275.$$

The decay constant for the strangeness-changing weak decays turns out to be smaller than that for the strangeness conserving decays. The Cabibbo theory of currents which we discuss next was put forward to explain this.

6.9 Cabibbo Theory of Currents

The above-mentioned features are all accounted for by the Cabibbo theory, which preserves the concept of a universal V-A coupling. In the language of the quark model, following Cabibbo, we redefine the hadronic weak current

$$J_\mu = \bar{u}\gamma_\mu(1 - \gamma_5)d \tag{6.192}$$

by a rotation in $SU(3)$ space, in which

$$\left.\begin{aligned} &u \rightarrow u, \\ &d \rightarrow d_c = d \cos\theta_c + s \sin\theta_c \\ &s \rightarrow s_c = s \cos\theta_c - d \sin\theta_c. \end{aligned}\right\} \tag{6.193}$$

The hadronic weak current is now:

$$J_\mu = \bar{u}\gamma_\mu(1-\gamma_5)d_c \qquad (6.194)$$

$$= \bar{u}\gamma_\mu(1-\gamma_5)d\cos\theta_c + \bar{u}\gamma_\mu(1-\gamma_5)s\sin\theta_c \qquad (6.195)$$

and is assumed to couple to the leptonic current with the universal strength G_F. Since we know that

$$\frac{\text{amp }(|\Delta S| = 1)}{\text{amp }(|\Delta S| = 0)} = 1/5, \qquad (6.196)$$

we need

$$\tan\theta_c = 0.2, \qquad \cos\theta_c = 0.98. \qquad (6.197)$$

An immediate consequence of this is a small modification of the effective β-decay coupling constant. This modification, in fact, improves the agreement between the couplings measured in μ-decay and β-decay. For the $|\Delta S| = 0$ decays the effective coupling is $G_F\cos\theta_c$, whereas for the $|\Delta S| = 1$ decays, it is $G_F\sin\theta_c$. The Cabibbo angle θ_c, therefore, can be determined experimentally by comparing the $|\Delta S| = 1$ amplitude with $|\Delta S| = 0$ amplitude. For example,

$$\frac{K^+ \to \mu^+\nu_\mu}{\pi^+ \to \mu^+\nu_\mu} \to \tan\theta_c = 0.275 \pm 0.003.$$

for axial current, and

$$\frac{K^+ \to \pi^0 e^+\nu_e}{\pi^+ \to \pi^0 e^+\nu_e} \to \tan\theta_c = 0.251 \pm 0.008.$$

for vector current.

6.10 Intermediate Vector Boson Hypothesis

In the intermediate vector boson hypothesis, the Hamiltonian density is taken as

$$\mathcal{H}_W = g_W J_\lambda(x) W^\lambda(x) + \text{h.c.}, \qquad (6.198)$$

where $W^\lambda(x)$ is the field of a charged vector boson which in the weak interactions is supposed to play the same role as played by photon in quantum electrodynamics (QED). The weak interactions are then supposed to be mediated by the charged intermediate vector boson (IVB). For example, the μ-decay can be expressed as in Fig. 6.10, and the neutron β-decay as in Fig. (6.11). This can be regarded as the first step towards an eventual unification of the weak and the electromagnetic interactions. The matrix element has now the form

$$M = g_W^2 J^\mu_{\text{weak}} \left[\frac{\delta_{\mu\nu} - (q_\mu q_\nu/M_W^2)}{q^2 - M_W^2}\right] J_\nu^{\text{weak}}. \qquad (6.199)$$

At low energies, e.g. at typical β-decay energies,

$$q^2 \ll M_W^2$$

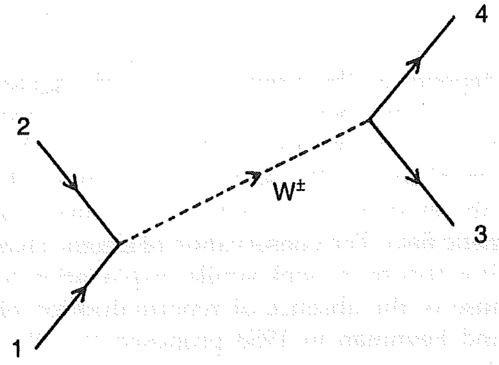

Fig. 6.10 Intermediate vector boson hypothesis.

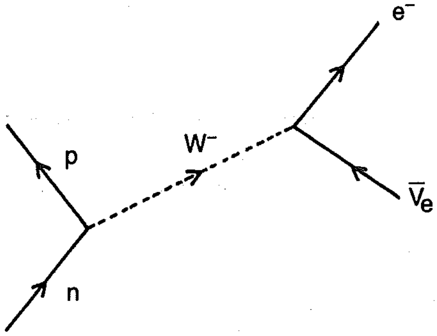

Fig. 6.11 Neutron decay through IVB.

so that q^2 dependence of the propagator can be ignored and we can write the qualitative equivalence

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{M_W^2}. \tag{6.200}$$

6.11 CVC and PCAC

6.11.1 CVC Hypothesis

One of the most remarkable results which came out of the investigations of μ-meson decays and beta decay was the near equality of the μ coupling constant G_F and the effective vector coupling constant $G_V(q^2 = 0)$ of beta decay. The explanation of this result came from Gerstein and Zeldovich[1] and Feynman and Gell-Mann,[2] in terms of the conserved vector current hypothesis.

In the electromagnetic interactions, the matrix element for the elastic electron-positron scattering can be calculated in the lowest order, whereas the electron-proton scattering involves the interplay of strong interaction and so depends on the form (structure) factor of proton which in turn depends on the momentum transfer square. Yet the matrix elements are equal in the limit of zero momentum transfer. This is because at zero momentum transfer, the matrix element is determined by the charges of the interacting particles and electric charge is not normalized by the strong interactions. This universality of electric charge can be traced to the existence of a conserved electromagnetic current which is the source of the electromagnetic field. The conservation of electric charge is intimately connected with gauge invariance. It is natural to seek similar explanation for the vector part of weak interaction current because of the absence of renormalization effects here too.

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[1] S.S. Gerstein and Ya.B. Zeldovich (1958): Soviet Physics, *JFTP,* **8**, 570.

[2] R.P. Feynman and M. Gell-Mann (1958): *Phys. Rev.*, **109**, 193.

Gerstein in 1956 and Feynman in 1958 proposed that the hadronic charged vector currents coupled to leptons are conserved isospin currents.

$$\partial^\mu V_\mu = 0, \qquad \partial^\mu V_\mu^\dagger = 0, \tag{6.201}$$

$$V_\mu = V_\mu^1 + iV_\mu^2 \equiv V_\mu^{(+)}, \qquad V_\mu^\dagger = V_\mu^1 - iV_\mu^2 \equiv V_\mu^{(-)}, \tag{6.202}$$

implying that there are conserved quantum numbers associated with the charges

$$Q^\pm = \int d^3x V_0^{(\pm)}(x), \tag{6.203}$$

which must be time independent, i.e.

$$\frac{\partial}{\partial x_0} Q(x_0) = \int d^3x \frac{\partial V_0}{\partial x_0} = \int d^3x\, \nabla \cdot V(x) = 0 \tag{6.204}$$

since the integral of $\nabla \cdot V$ vanishes identically due to the boundary condition at infinity. The charges are, therefore, invariant with respect to strong interactions. These charges are just the isospin raising and lowering operators:

$$I_\pm = I_1 \pm iI_2 \tag{6.205}$$

$$I_i = \int d^3x V_0^i(x), \qquad i = 1, 2, 3. \tag{6.206}$$

The identification of the weak current implies that the semi-leptonic processes satisfy the selection rules

$$\Delta Q = \Delta I_3 = \pm 1, \qquad |\Delta I| = 1.$$

Moreover, (a) the charge-raising and charge-lowering currents are the members of the same triplet of isospin, i.e.

$$e^{i\pi I_2}\, V_\mu^{(+)}\, e^{i\pi I_2} = -\, V_\mu^{(-)}. \tag{6.207}$$

(b) The neutral member of this triplet, V_μ^3, is just the isovector part of the conserved electromagnetic current. If we write

$$j_\mu^{\text{em}} = V_\mu^0 + V_\mu^3, \tag{6.208}$$

then

$$[I_\pm, V_\mu^0] = 0, \qquad [I_\pm, V_\mu^3] = \mp V_\mu^3. \tag{6.209}$$

Historically, CVC was the first hypothesis linking electromagnetic observables with weak observables.

6.11.2 Experimental Verification of CVC

There exist several experimental verifications of isotriplet vector current hypothesis, as also of CVC.

Pion-β decay

$$\pi^\pm \to \pi^0 + e^\pm + \nu_e(\overline{\nu}_e).$$

From the isotriplet vector current hypothesis, the matrix elements of

$$\pi^\pm \to \pi^0 + e^\pm + \nu_e(\overline{\nu}_e), \qquad \pi^\pm \to \pi^\pm + \gamma$$

are related. Neglecting four momentum transfer dependence, one can calculate the branching ratio

$$\frac{\text{Rate }(\pi^\pm \to \pi^0 e^\pm \nu)}{\text{Rate }(\pi^\pm \to \pi^\pm \gamma)} = 1.07 \times 10^{-8}, \tag{6.210}$$

which is in good agreement with the experimental value

$$(1.02 \pm .07) \times 10^{-8}.$$

The hadronic matrix element is given by

$$\langle \pi^0(p_f) | j_\mu^{\text{had}}(x) | \pi^-(p_i) \rangle = e^{-iq\cdot x}[P_\mu f_+ + q_\mu f_-], \tag{6.211}$$

where

$$P_\mu = (p_i + p_f)_\mu, \qquad q_\mu = (p_i - p_f)_\mu. \tag{6.212}$$

The form factors $f_\pm$ are Lorentz invariant and therefore depend on the Lorentz scalars q^2, $q \cdot P$ and P^2. However, only one of these is independent since

$$q \cdot P = m_{\pi^-}^2 - m_{\pi^0}^2, \qquad P^2 + q^2 = 4m_\pi^2. \tag{6.213}$$

We take q^2 as the independent variable. Due to momentum conservation,

$$q = p_e + p_\nu. \tag{6.214}$$

Now the current conservation implies

$$\partial^\mu \langle \pi^0 | V_\mu^{(\pm)}(x) | \pi^- \rangle = iq^\mu \langle \pi^0 | V_\mu^{(\pm)}(x) | \pi^- \rangle = 0. \tag{6.215}$$

From the definition of the form factors,

$$q \cdot P f_+(q^2) + q^2 f_-(q^2) = 0. \tag{6.216}$$

However, $q \cdot P = m_{\pi^-}^2 - m_{\pi^0}^2$, which is zero if we ignore small electromagnetic mass difference. So we have

$$f_-(q^2) = 0. \tag{6.217}$$

This prediction is only academic as f_- does not contribute significantly to the pion decay as can be seen from the leptonic part of the matrix element. The important prediction is the normalization of f_+. The matrix element of the charge associated with $V_\mu(x)$ is

$$\int d^3x \langle \pi^0 | V_0(x) | \pi^- \rangle = \int d^3x e^{-iq \cdot x} P_0 f_+(q^2) = (2\pi)^3 \delta(\mathbf{q}) P_0 f_+(0) \tag{6.218}$$

since for $\mathbf{q} = 0$, $q_0 = E_- - E_0 = 0$. The isospin raising operator has the property

$$\langle \pi^0(p_f) | I_+ | \pi^-(p_i) \rangle = \sqrt{2} \langle \pi^0(p_f) | \pi^0(p_i) \rangle = \sqrt{2} E_\pi (2\pi)^3 \delta(p_f - p_i), \tag{6.219}$$

where we have used the normalization

$$\langle p' | p \rangle = 2E(2\pi)^3 \delta(p' - p).$$

Comparing (6.218) with (6.219), we obtain

$$f_+ = \sqrt{2}. \tag{6.220}$$

The pion matrix element is now completely determined and we can obtain a prediction for the rate (including radiative corrections).

6.11.3 Lack of Conservation of Axial Vector Current

In view of the discussion on V_μ, a spontaneous question arises, i.e. whether the axial vector current A_μ is also conserved. The ratio $G_A/G_V \sim 1.25$ implies that it probably is not. The axial vector coupling constant is renormalized in contrast to the vector current and so is indicative of the non-conservation of the axial current. But there are stronger arguments. Consider the pion decay

$$\pi \to l + \nu_l.$$

The amplitude for the decay involves the matrix element of the axial vector current between the vacuum and a pion state with momentum p, i.e.

$$\langle 0 | A_\mu^+(x) | \pi^-(p) \rangle = i\sqrt{2} f_\pi p_\mu \tag{6.221}$$

which is in direct contrast with the conservation of A_μ. In fact, we evaluate the vacuum-one pion matrix element of $\partial^\lambda A_\lambda$

$$\langle 0 | \frac{\partial A_\lambda^+}{\partial x_\lambda} | \pi^-(p) \rangle = i \langle 0 | [P^\lambda, A_\lambda^+] | \pi^- \rangle, \tag{6.222}$$

$$-iq^\lambda \langle 0 | A_\lambda^+(x) | \pi^-(p) \rangle = m_\pi^2 f_\pi. \tag{6.223}$$

This is not zero; otherwise, f_π would be zero and decay mode π_{l2} would not be possible. Consequently,

$$\frac{\partial A^\lambda}{\partial x_\lambda} \neq 0. \tag{6.224}$$

However, we notice that whatever the actual value of the pion decay constant f_π, the right-hand side of equation (6.224) would vanish if the pion mass were zero. This leads to the hypothesis that $\partial^\mu A_\mu$, although certainly not zero, is still small in the sense that its matrix element would vanish if $m_\pi^2 \to 0$. Furthermore, $\partial^\mu A_\mu(x)$ is a pseudoscalar operator with unit isospin, zero hypercharge and odd G parity and it has all the quantum numbers as the pion triple field. It appears that the simplest ansatz is

$$\partial^\mu A_\mu = C_\pi \phi(x), \tag{6.225}$$

where C_π is a constant. Equation (6.225) is usually called the *Partially Conserved Axial-vector Current* (PCAC) hypothesis. The essential idea of PCAC is that this operator is a good one for use in describing the creation and destruction of pions. The presumed goodness of the axial vector divergence is supposed to mean that the matrix elements of this operator are very slowly varying in the mass variable as one goes a little way offshell. In other words, the equation is to be understood as the definition of an interpolating field for the pion.

We can relate the constant C_π to the pion decay constant. Let us suppose the pion field operator ϕ^i to be normalized according to

$$\langle 0 | \phi^i | \pi^j \rangle = \delta_{ij}. \tag{6.226}$$

We make the identification

$$\frac{1}{m_\pi^2 f_\pi} \frac{\partial A_\mu^i}{\partial x_\mu} \equiv \phi^i \tag{6.227}$$

so that

$$C_\pi = \frac{m_\pi^2 f_\pi}{\sqrt{2}}. \tag{6.228}$$

6.12 Goldberger Treiman Relation

Let us now consider the matrix element of the axial vector current between nucleon states

$$\langle N(p_2) | A_\mu^\alpha(x) | N(p_1) \rangle = \bar{u}(p_2) \, [\gamma_\mu G_1(q^2) + q_\mu G_2(q^2)] \gamma_5 \frac{1}{2} \tau^\alpha u(p_1), \tag{6.229}$$

where $q = p_2 - p_1$ and $G_1(q^2)$ is normalized with

$$G_1(0) = 1.25. \tag{6.230}$$

The matrix element of the divergence $\overline{D}^\alpha \equiv \partial^\mu A_\mu^\alpha$

$$\langle N(p_2)|\partial^\mu A^\alpha_\mu(x)|N(p_1)\rangle \equiv iq^\mu\langle N(p_2)|A^\alpha_\mu(x)|N(p_1)\rangle$$

$$= iq^\mu\,\overline{u}(p_2)\,[\gamma_\mu G_1(q^2) + q_\mu G_2(q^2)]\gamma_5\,\frac{1}{2}\tau^\alpha u(p_1) \tag{6.231}$$

$$= i\overline{u}(p_2)\gamma_5\,\frac{1}{2}\tau^\alpha u(p_1)G_D(q^2), \tag{6.232}$$

where

$$G_D(q^2) = 2M_N G_1(q^2) + q^2 G_2(q^2). \tag{6.233}$$

Exercise Prove that the exact conservation of axial vector current implies $G_D = 0$.

Solution One way to find the solution is to require at the same time

$$M_N = 0, \qquad G_2(q^2) = 0. \tag{6.234}$$

Since these conditions are far from being satisfied in the real world, we require that the axial vector current conservation follows from a cancellation between G_1 and G_2, as given by

$$2M_N G_1(q^2) + q^2 G_2(q^2) = 0. \tag{6.235}$$

However, if we set $q^2 = 0$, this would seem to lead to $G_1(q^2) = 0$ again in contradiction with experiment. The way out is to let $G_2(q^2)$ have a pole at $q^2 = 0$ due to a massless particle (i.e. pion) so that $\lim\limits_{q^2\to 0} q^2 G_2(q^2)$ does not vanish.

$$G_1(0) = -\frac{1}{2M_N}\lim_{q^2\to 0} q^2 G_2(q^2). \tag{6.236}$$

If we consider the vertex $\langle N(p_2)\,|\,A^\alpha_\mu(x)\,|\,N(p_1)\rangle$ and assume the contribution to come from the exchange of a pion of zero mass (Fig. 6.12),

$$-iq_\mu f_\pi 2g_{\pi NN} i\overline{u}_2\gamma_5\,\frac{1}{2}\,\tau_\alpha u_1\,\frac{1}{-q^2} \tag{6.237}$$

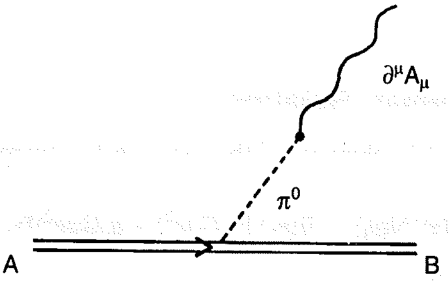

Fig. 6.12 Pion exchange.

so that

$$[G_2(q^2)]_\pi = 2g_{\pi NN}\,\frac{f_\pi}{-q^2}, \tag{6.238}$$

and we immediately have

$$G_1(0) = \frac{g_{\pi NN} f_\pi}{M_N}. \tag{6.239}$$

Adler consistency condition

Let us consider, in general, the consequences of current conservation in the $q_\mu \to 0$ limit. Let

$$T_\mu = \langle b \,|\, A_\mu^\alpha(x) \,|\, a \rangle, \tag{6.240}$$

where $|a\rangle$ and $|b\rangle$ are arbitrary hadron states. The current conservation implies

$$q^\mu T_\mu = 0, \tag{6.241}$$

where $q = p_b - p_a$. Let us write

$$T_\mu = (T_\mu)_\pi + \overline{T}_\mu, \tag{6.242}$$

where $\{T_\mu\}_\pi$ is the zero mass pion pole contribution defined as

$$(T_\mu)_\pi = -i \frac{f_\pi q_\mu}{-q^2} G_{\pi a \to b}(q^2). \tag{6.243}$$

So

$$G_{\pi a \to b} = \frac{i}{f_\pi} q^\mu \overline{T}_\mu. \tag{6.244}$$

6.13 Current Commutators

Let us consider a set of n Dirac field operators; ψ_α $(\alpha = 1, 2, \ldots, n)$. The canonical commutation relations are

$$\{\psi_\alpha(x), \psi_\beta(x')\}_{x_0 = x'_0} = 0 = \{\psi^{+\alpha}(x), \psi^{+\beta}(x')\}, \tag{6.245}$$

$$\{\psi_\alpha(x), \psi^{+\beta}(x')\}_{x_0 = x'_0} = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{x}'). \tag{6.246}$$

Using the identity

$$[A, BC] \equiv \{A, B\}C - B\{A, C\}, \tag{6.247}$$

and the above canonical anticommutation rules, we can prove that

$$[\psi^{+\alpha}(x)\,\psi_\beta(x), \psi^{+\mu}(x')\,\psi_\nu(x')]_{x_0 = x'_0} = [\delta^\mu_\beta\,\psi^{+\alpha}(x)\,\psi_\nu(x) - \delta^\alpha_\nu\,\psi^{+\mu}(x)\,\psi_\beta(x)] \times \delta(\mathbf{x} - \mathbf{x}'). \tag{6.248}$$

This is a very useful relation. With the help of this relation we can immediately write down the Lie algebra of unitary groups. From (6.248) we can immediately see that

$$[\psi^+(x)A\psi(x), \psi^+(x')B\psi(x')] = \psi^+(x)[A, B]\psi(x')\delta(\mathbf{x} - \mathbf{x}'). \tag{6.249}$$

In general, we may write

$$A \equiv A_1 \otimes A_2, \qquad B \equiv B_1 \otimes B_2, \tag{6.250}$$

where the operators with the subscript 1 correspond to the Dirac matrices, whereas the subscript 2 to the internal degree of freedom and so correspond to the isospin matrices, Gell-Mann's λ matrices of $SU(3)$ or the $SU(n)$ matrices in general. In other words, the operators A_1 and B_1 act on the Lorentz space, whereas the operators A_2 and B_2 act on the internal unitary space. The operation $\otimes$ stands for direct product.

We can then see that

$$[A, B] = \frac{1}{2}[A_1, B_1]\{A_2, B_2\} + \frac{1}{2}\{A_1, B_1\}[A_2, B_2]. \tag{6.251}$$

We make use of (6.249) and (6.251) to derive the commutation relations among the generators of the isospin transformations. We have seen earlier that

$$I_i \equiv Q_i = \int d^3x \bar{\psi}(x)\gamma_0 \frac{1}{2}\tau_i\psi(x) = \int d^3x \psi^{\dagger}(x)\frac{1}{2}\tau_i\psi(x). \tag{6.252}$$

So

$$\begin{aligned}[Q_i, Q_j] &= \left[\int d^3x\psi^{\dagger}(x)\frac{1}{2}\tau_i\psi(x), \int d^3x'\psi^{\dagger}(x')\frac{1}{2}\tau_j\psi(x')\right] \\ &= \int d^3x \int d^3x'\psi^{\dagger}(x)\left[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j\right]\psi(x)\delta(\mathbf{x}-\mathbf{x}') \\ &= i\varepsilon_{ijk}\int d^3x\psi^{\dagger}(x)\frac{1}{2}\tau_k\psi(x) = i\varepsilon_{ijk}Q_k\end{aligned} \tag{6.253}$$

Let us now define axial charge Q_i^5 as the integral of the time component of the axial vector current, namely,

$$Q_i^5 = \int d^3x\bar{\psi}(x)\gamma_0\gamma_5\frac{1}{2}\tau_i\psi(x), \tag{6.254}$$

$$\begin{aligned}[Q_i, Q_j^5] &= \left[\int d^3x\bar{\psi}(x)\gamma_0\frac{1}{2}\tau_i\psi(x), \int d^3x\bar{\psi}(x)\gamma_0\gamma_5\frac{1}{2}\tau_j\psi(x)\right] \\ &= \left[\int d^3x\psi^{\dagger}(x)\frac{1}{2}\tau_i\psi(x), \int d^3x\psi^{\dagger}(x)\gamma_5\frac{1}{2}\tau_j\psi(x)\right] \\ &= \int d^3x\int d^3x'\psi^{\dagger}(x)\left[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j\gamma_5\right]\psi(x)\delta(\mathbf{x}-\mathbf{x}') \\ &= i\varepsilon_{ijk}\int d^3x\bar{\psi}(x)\gamma_0\gamma_5\frac{1}{2}\tau_k\psi(x) = i\varepsilon_{ijk}Q_k^5,\end{aligned} \tag{6.255}$$

where, in the derivation of (6.254), we have made use of relations (6.249) and (6.251).

Equation (6.255) is the statement of the fact that the axial-vector current $\bar{\psi}\gamma_\mu\gamma_5\frac{1}{2}\tau_i\psi$ and the axial charge Q_i^5 transforms like an isovector. Similarly, we can show that

$$[Q_i^5, Q_j^5] = i\varepsilon_{ijk}Q_k. \tag{6.256}$$

The commutation relations (6.253), (6.255) and (6.256) then close under the commutation rules for Q_i and Q_i^5 and constitute a Lie Algebra for the chiral group $SU(2) \otimes SU(2)$.

Exercises **1.** Show that

$$F(q) = \int d^3r\rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}.$$

Solution

$$V(\mathbf{r}) = e\Phi(\mathbf{r}) = e\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

$$\nabla^2\Phi = -\rho(\mathbf{r}).$$

The Fourier transform of the potential energy V is given by

$$\int d^3r\, V(r)\, e^{i\mathbf{q}\cdot\mathbf{r}} = e\left(-\frac{1}{q^2}\right)\int d^3r\, \nabla^2(e^{i\mathbf{q}\cdot\mathbf{r}})\Phi(r) = -e\frac{1}{q^2}\int d^3e^{i\mathbf{q}\cdot\mathbf{r}}\nabla^2\,\Phi(r)$$

$$= e\frac{1}{q^2}\int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}}\rho(r) = e\frac{1}{q^2}F(q). \tag{6.257}$$

2. Derive the form factor for a homogeneous charge density,

$$\rho(r) = \frac{3}{4\pi R^3}\,\theta(R-r).$$

Solution

$$F(q^2) = \frac{3}{Z}j_1(Z) = \frac{3}{Z^3} = [\sin Z - Z\cos Z],$$

where $Z = qR$.

3. Obtain an expression for the decay rate for $\pi^- \to l + \nu_l$.

6.13.1 Solutions

Weak interactions

Using the current × current picture of the weak interaction and the Lorentz covariance for the matrix element of the weak hadronic current between the vacuum and a single pion state, we can write the matrix element of the decay

$$\pi \to l + \bar{\nu}_l$$

as

$$M = \frac{G}{\sqrt{2}}(p'^{\mu} + q^{\mu}) f_\pi \bar{u}(p')\gamma_\mu (1 - \gamma_5) v(q)$$

$$= \frac{G}{\sqrt{2}} p'^{\mu} f_\pi \bar{u}(p')\gamma_\mu (1 - \gamma_5) v(q)$$

because

$$\gamma \cdot q v(q) = 0.$$

Further, $\bar{u}(p')(\gamma \, . \, p' - m_\mu) = 0$ gives us

$$M = \frac{G}{\sqrt{2}} m_\mu f_\pi \bar{u}(p)(1 - \gamma_5) v(q) = \frac{G}{\sqrt{2}} m_\mu f_\pi u^\dagger(p')\gamma_0 (1 - \gamma_5) v(q)$$

$$= \frac{G}{\sqrt{2}} m_\mu f_\pi u^\dagger(p')(1 + \gamma_5)\gamma_0 v(q).$$

So,

$$M^\dagger = \frac{G}{\sqrt{2}} m_\mu f_\pi v^\dagger(q)\gamma_0 (1 + \gamma_5) u(p') = \frac{G}{\sqrt{2}} m_\mu f_\pi \bar{v}(q)(1 + \gamma_5) u(p). \qquad (6.258)$$

The sum over the final spins gives

$$\sum_{\text{spin}} MM^\dagger = \frac{G^2}{2} m_\mu^2 f_\pi^2 \sum_{\text{spin}} \bar{u}(p')(1 - \gamma_5) v(q) \times \bar{v}(q)(1 + \gamma_5) u(p)$$

$$= \frac{G^2}{2} m_\mu^2 f_\pi^2 Tr[(\gamma \cdot p' + m_\mu)(1 - \gamma_5)(\gamma \cdot q)(1 + \gamma_5)]$$

$$= G^2 m_\mu^2 f_\pi^2 Tr[(\gamma \cdot p' + m_\mu)(\gamma \cdot q)(1 + \gamma_5)]$$

$$= G^2 m_\mu^2 f_\pi^2 \, 4p' \cdot q.$$

Substituting this into the formula for the decay rate (see Appendix E), we get

$$\Gamma = \frac{1}{32\pi^2 m_\pi^2} 4\pi |\mathbf{q}| \, G^2 m_\mu^2 f_\pi^2 \, 4p' \cdot q = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 .$$

We now obtain

$$|\mathbf{q}| = \frac{1}{2m_\pi}(m_\pi^2 - m_\mu^2), \qquad q \cdot p' = \frac{1}{2}(m_\pi^2 - m_\mu^2),$$

and

$$\frac{\Gamma(\pi \to e^- \bar{\nu}_e)}{\Gamma(\pi \to \mu^- \bar{\nu}_\mu)} = \frac{m_e^2 (m_\pi^2 - m_e^2)^2}{m_\mu^2 (m_\pi^2 - m_\mu^2)^2} = 1.2 \times 10^{-4}.$$

PROBLEMS

Electromagnetic interactions

1. Explain what you understand by a form factor. Obtain the expression for the electromagnetic form factor in terms of charge density distribution.
2. For a spherically symmetric uniform charge distribution

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases}$$

show that the form factor is given by

$$F(q^2) = \frac{4\pi\rho_0}{q^3}\,[\sin(qa) - qa\cos(qa)].$$

3. Show that for a charge distribution,

$$\rho(r) = \frac{m^3}{8\pi}e^{-mr}, \qquad F(q^2) = \left(1 + \frac{q^2}{m^2}\right)^{-2}.$$

4. Show that the most general form of the matrix element of electromagnetic current between two proton states is

$$\langle p' | j_\mu | p \rangle = \bar{u}(p')[F_1(q^2)\gamma_\mu + F_2(q^2)\sigma_{\mu\nu}q^\nu].$$

Weak interactions

5. For each of the following sets of decays, state the quark composition of the particles involved and draw a diagram to show decays in terms of quarks and leptons. Ignoring kinematical factors, find the ratio of the rates of the decays in each set in terms of the Cabibbo angle.

(i) $n \to p + e^- + \bar{\nu}_e$, $\Lambda \to p + e^- + \bar{\nu}_e$.
(ii) $\Sigma^- \to \Lambda + e^- + \bar{\nu}_e$, $Xi^- \to \Lambda + e^- + \bar{\nu}_e$.
(iii) $\Lambda_c \to \Lambda + e^+ + \nu_e$, $\Lambda_c \to n + e^+ + \nu_e$.
(iv) $K^- \to \pi^0 + e^- + \nu_e$, $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$.
(v) $D^0 \to K^- + e^+ + \nu_e$, $D^0 \to \pi^- + e^+ + \nu_e$.
(vi) $D^0 \to K^- + \pi^0 + \pi^+$, $D^0 \to K^- + K^0 + \pi^+$.
(vii) $D^0 \to \pi^- + \pi^0 + \pi^+$, $D^0 \to \pi^- + K^0 + \pi^+$.
(viii) $D_s^+ \to K^0 + \pi^+$, $D_s^+ \to K^0 + K^+$.
(ix) $D_s^+ \to \eta^0 + \pi^+$, $D_s^+ \to \eta^0 + K^+$.
(x) $\pi^- \to \mu^- + \bar{\nu}_\mu$, $K^- \to \mu^- + \bar{\nu}_\mu$.
(xi) $D^0 \to K^- + \pi^+$, $D^0 \to K^- + K^+$.
(xii) $D^0 \to \bar{\pi} + \pi^+$, $D^0 \to \pi^- + K^+$.

6. Compute the number of real parameters of an $n \times n$ orthogonal matrix and the number of independent parameters of an $n \times n$ unitary matrix. Then obtain the residual phases. Give the physical significance of the phase in the case of $n = 3$.

7

NONLEPTONIC WEAK INTERACTION

7.1 Nonleptonic Decays

In studying the semi-leptonic reactions we could reasonably assume an interaction Hamiltonian of current ⊗ current form, and to the lowest order the leptonic part of the transition could then be easily computed. The strong interaction effects could be isolated in the hadronic currents and their matrix elements. For the nonleptonic decays, where the participating particles are strongly interacting, such factorization is not meaningful. Here one runs into dynamical problems at an early stage. Nevertheless, we can ask general type of questions. What are the space time properties of the nonleptonic reactions? What are the internal symmetry properties: *SU*(2) or *SU*(3) properties etc.?

7.2 Nonleptonic Weak Decays of Strange Particles

We discuss the hyperon decays:

$$\left.\begin{aligned} &\Lambda \to N + \pi, \\ &\Sigma \to N + \pi, \\ &\Xi \to \Lambda + \pi, \\ &\Omega \to \Xi + \pi,\ K + \Lambda,\ \Xi^* + \pi. \end{aligned}\right\} \tag{7.1}$$

We shall discuss invariance (or lack of invariance) with respect to S, I, I_3, P, T and C. The quantum numbers S, I, I_3, P and C are all conserved in strong interactions but not in weak interactions. Time reversal T is assumed to hold in both the strong and weak interactions. There is no experimental evidence to the contrary.

7.2.1 Space Time Properties

Let us, as an example, consider the Λ decay. The transition matrix element can be written in Pauli spin space for the Λ hyperon rest frame as

$$M = \chi_p^\dagger (S - P\sigma\cdot\hat{\mathbf{p}})\chi_\Lambda \tag{7.2}$$

where S and P are the amplitudes for decay into S- and P-wave channels and $\hat{\mathbf{p}}$ is a unit vector in the direction of the decay nucleon momentum. For the polarized Λ, the angular distribution of the decay proton is given by

$$w(\theta)\,d\cos\theta = \frac{1}{2}(1 - \alpha\hat{\mathbf{p}}\cdot\Sigma)\,d\cos\theta, \tag{7.3}$$

where Σ is the polarization of Λ and α, the asymmetry parameter given by

$$\alpha = \frac{2\,\mathrm{Re}\,(S^* P)}{|S|^2 + |P|^2}.$$

The information on the other asymmetry parameters β and γ defined by

$$\beta = \frac{2\,\mathrm{Im}\,(S^* P)}{|S|^2 + |P|^2}, \qquad \gamma = \frac{|S|^2 - |P|^2}{|S|^2 + |P|^2}$$

is obtained by measuring the polarization of the decay nucleon. These three asymmetry parameters along with the decay rate then determine the S- and P-wave amplitudes.

7.2.2 Isospin Properties

To illustrate the non-conservation of S, I, I_3, for example, in weak interactions, we consider

$$\Lambda \to p + \pi^-.$$

None of the S, I, I_3 is conserved. Strangeness for Λ is -1, whereas for both p and π it is 0. So $\Delta S = S_f - S_i = 0 - (-1) = 1$. The value of isospin I is zero for the initial state Λ. For the final state (nucleon-pion state), the isospin can be either 1/2 or 3/2 so that $\Delta I = 1/2$ or 3/2. We note that

$$\Delta I_3 = -1/2.$$

7.2.3 $\Delta I = 1/2$ Rule for Nonleptonic Decays

Gell-Mann and Rosenfeld in 1957[1] assumed minimum violation in isospin, i.e. $\Delta I = 1/2$ in nonleptonic weak interactions. This implies that the relevant weak interaction Hamiltonian transforms like the component of an isospinor. To see the consequences of the $\Delta I = 1/2$ rule, let us consider a specific example of the decay modes of Λ.

$$\Lambda \to p + \pi^-, \qquad \Lambda \to n + \pi^0.$$

The initial state has $I = 0$; hence the rule implies that the final state will be in $I = 1/2$ state. With the help of CG coefficients, we see that

$$\left.\begin{aligned} |\pi^- p\rangle &= \sqrt{\frac{1}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \\ |\pi^0 n\rangle &= \sqrt{\frac{2}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle. \end{aligned}\right\} \tag{7.4}$$

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[1] M. Gell-Mann and A.H. Rosenfeld (1957): *Ann. Rev. Nucl. Sci.*, 7, 407.

So the decay amplitude for the decay $\Lambda \to p + \pi^-$ can be written as

$$\langle p\pi^- | H_W | \Lambda\rangle = \sqrt{\frac{1}{3}}\langle I = \frac{3}{2} \| H_W \| \Lambda\rangle + \sqrt{\frac{2}{3}}\langle I = \frac{1}{2} \| H_W \| \Lambda\rangle \tag{7.5}$$

Similarly,

$$\langle n\pi^0 | H_W | \Lambda\rangle = \sqrt{\frac{2}{3}}\langle I = \frac{3}{2} \| H_W \| \Lambda\rangle + \sqrt{\frac{1}{3}}\langle I = \frac{1}{2} \| H_W \| \Lambda\rangle. \tag{7.6}$$

This immediately gives us the following equation which is due to $\Delta I = 1/2$ rule.

$$\frac{\text{amp}\ (\Lambda \to p\pi^-)}{\text{amp}\ (\Lambda \to n\pi^0)} = -\sqrt{2}. \tag{7.7}$$

Alternatively, one can obtain this ratio with the help of the so-called *spurion technique*. It is convenient to introduce a spurion S (a hypothetical particle which does not carry any energy, momentum or any other conserved quantum number) to keep track of the change of isospin and flavour in the decay. The decay $\Lambda \to p + \pi^-$ can be thought of as a reaction $S + \Lambda \to p + \pi^-$ in which all quantities are conserved. The violations of the decay being taken care of by the spurion, we can think of the spurion carrying the ΔS and ΔI. The spurion, thus, will transform under isospin like a member with $I = 1/2$ and $I_3 = -1/2$. The Hamiltonian describing the decay can then be written in the isospin invariant form

$$H_W = \overline{N}^a \pi_a^b S_b \Lambda = \mathrm{Tr}[\overline{N}\pi S]\Lambda, \tag{7.8}$$

where the symbols stand for the fields corresponding to the particle

$$S = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

i.e. like the iso-spin down state. So,

$$H_W = a\overline{N}^a \pi_a^2 \Lambda = a\overline{N}^1 \pi_1^2 \Lambda + a\overline{N}^2 \pi_2^2 \Lambda = a\bar{p}\pi^+ \Lambda + a\bar{n}\left(-\frac{\pi^0}{\sqrt{2}}\right)\Lambda. \tag{7.9}$$

Now the amplitude for the decay $\Lambda \to p + \pi^-$, which is the coefficient of the term $\bar{p}\pi^+ \Lambda$ in the Hamiltonian density (7.9) representing the decay, is given by

$$\text{amp}\ (\Lambda \to p + \pi^-) = a. \tag{7.10}$$

Similarly,

$$\text{amp}\ (\Lambda \to n + \pi^0) = -\frac{1}{\sqrt{2}}a$$

so that we have

$$\text{amp}\ (\Lambda \to p + \pi^-) + \sqrt{2}\ \text{amp}\ (\Lambda \to n + \pi^0) = 0 \tag{7.11}$$

where "amp" stands for the parity violating amplitude A or the parity conserving amplitude B. The branching ratio is

$$\frac{\Gamma(\Lambda \to p\pi^-)}{\Gamma(\Lambda \to n\pi^0)} = 2 \tag{7.12}$$

or the branching fraction is

$$\frac{\Gamma(\Lambda \to p\pi^-)}{\Gamma(\Lambda \to n\pi^0) + \Gamma(\Lambda \to p\pi^-)} = \frac{2}{3}. \tag{7.13}$$

Experimentally this number is 64.2 ± 0.5%, which is in excellent agreement with the $\Delta I = 1/2$ rule value.

7.2.4 The Decay $K \to 2\pi$

We now consider the decays

$$K^+ \to \pi^+ + \pi^0, \qquad K^0 \to \pi^+ + \pi^-, \qquad K^0 \to \pi^0 + \pi^0.$$

Two pions have total $I = 0, 1, 2$. We can use the Clebsch-Gordan coefficients to construct the charge states for 2π with $I = 0, 1,$ or 2. We now have

$$\left.\begin{aligned} |0,0\rangle &= \sqrt{\frac{1}{3}}\,[\pi^+\pi^- - \pi^0\pi^0 + \pi^-\pi^+], \\ |1,0\rangle &= \sqrt{\frac{1}{2}}\,[\pi^+\pi^- - \pi^-\pi^+], \\ |2,0\rangle &= \sqrt{\frac{1}{6}}\,[\pi^+\pi^- + 2\pi^0\pi^0 + \pi^-\pi^+]. \end{aligned}\right\} \tag{7.14}$$

The wave function of two pions is symmetric for $I = 0$, and 2 and antisymmetric for $I = 1$ with respect to the interchanging of the charge of 1 and 2. Since the two pions are identical bosons, the total exchange of space, spin and charge must leave the wave function unchanged. The pions have no spin, and l will be zero in the decay under consideration. For the K-decay, therefore, $I = 0$ or 2. $I = 0$ is excluded for K^+ by charge conservation, so it can go to 2π only in the state $I = 2$. However, K^0 can go to either $I = 2$ or $I = 0$ state. Thus, if $\Delta I = 1/2$ is strictly obeyed, K^+ decay would be forbidden but K^0 decay would be allowed. The experimental rate $R(K^+ \to \pi^+\pi^0)$ is only about 1/600 of $R(K_1^0 \to 2\pi)$ consistent with the $\Delta I = 1/2$ expectation.

Exercise Using the $\Delta I = 1/2$ rule, find a relation between the amplitudes a_+, a_- and a_0 for the decays $\Sigma^+ \to n\pi^+$, $\Sigma^- \to n\pi^-$ and $\Sigma^+ \to p\pi^0$, respectively.

Solution The weak interaction Hamiltonian responsible for these decays would have to be an $SU(2)$ spinor constructed out of the $SU(2)$ tensors Σ, π and the spinor N. There are two ways in which it can be done. So, we may write

$$\begin{aligned} \mathcal{H}_W^{NL} &= A\Sigma_b^c \pi_c^a \overline{N}^b S_a + B\Sigma_c^a \pi_b^c \overline{N}^b S_a = A\Sigma_b^c \pi_c^2 \overline{N}^b + B\Sigma_c^2 \pi_b^c \overline{N}^b \\ &= (A\Sigma_1^c \pi_c^2 + B\Sigma_c^2 \pi_1^c)\overline{p} + (A\Sigma_2^c \pi_c^2 + B\Sigma_c^2 \pi_2^c)\overline{n} \end{aligned}$$

with the value 2 for a. This gives

$$\left.\begin{aligned} \text{amp}\ (\Sigma^+ \to n\pi^+) &\equiv a_+ = B, \\ \text{amp}\ (\Sigma^- \to n\pi^-) &\equiv a_- = A, \\ \text{amp}\ (\Sigma^+ \to p\pi^0) &\equiv a_0 = \frac{1}{\sqrt{2}}(-A + B). \end{aligned}\right\} \tag{7.15}$$

Since these relations are true for both the S- and P-wave modes, we obtain the following triangular relation for the two-dimensional vectors (with S- and P-waves as two components along the x and y directions respectively):

$$a_+ + \sqrt{2}\,a_0 = a_- \tag{7.16}$$

Experimentally, $|a_+| \approx |a_0| \approx |a_-|$, so that the triangle is right angled. Also, the measurements on asymmetries obtained through determination of the polarization of final nucleon with polarized initial Σ give the results that the decay mode $\Sigma^+ \to n\pi^+$ is purely P-wave and the mode $\Sigma^- \to n\pi^-$ is purely S-wave. The data are consistent with the $\Delta I = 1/2$ rule.

7.3 Neutral *K*-Mesons

The particle K^0 and its antiparticle $\overline{K}^0$ are distinguished by strangeness quantum number. In decay interactions, however, strangeness is not a good quantum number and K^0 and $\overline{K}^0$ can transform into one another through second order weak interactions, e.g. $K^0 \to 2\pi \to \overline{K}^0$. This means that neither K^0 nor $\overline{K}^0$ have a definite life time. According to Gell-Mann and Pais (1954), it is convenient to use a set different from K^0 and $\overline{K}^0$ to describe the neutral K-meson decays. We define

$$\left.\begin{aligned} |K_1^0\rangle &= \frac{1}{\sqrt{2}}\{|K^0\rangle + CP\,|K^0\rangle\}, \\ |K_2^0\rangle &= \frac{1}{\sqrt{2}}\{|K^0\rangle - CP\,|K^0\rangle\}. \end{aligned}\right\} \tag{7.17}$$

Note that

$$CP\,|K^0\rangle = \eta\,|\overline{K}^0\rangle, \qquad |\eta| = 1\,. \tag{7.18}$$

We can adjust the phase of $|K^0\rangle$ so that $\eta = 1$. Then

$$\left.\begin{aligned} |K_1^0\rangle &= \frac{1}{\sqrt{2}}\{|K^0\rangle + |\overline{K}^0\rangle\}, \\ |K_2^0\rangle &= \frac{1}{\sqrt{2}}\{|K^0\rangle - |\overline{K}^0\rangle\}, \end{aligned}\right\} \tag{7.19}$$

and the inverse relations

$$|K^0\rangle = \frac{1}{\sqrt{2}}\{|K_1^0\rangle + |K_2^0\rangle\}, \qquad |\overline{K}^0\rangle = \frac{1}{\sqrt{2}}\{|K_1^0\rangle - |K_2^0\rangle\}. \tag{7.20}$$

$|K^0\rangle$ and $|\bar{K}^0\rangle$ are the eigenstates of strangeness with eigenvalues +1 and −1 respectively, but not the eigenstates of *CP*. $|K_1^0\rangle$ and $|K_2^0\rangle$ are the eigenstates of *CP* with eigenvalues +1 and −1 respectively, but not the eigenstates of strangeness *S*. Strong interactions conserve strangeness, and so neutral *K*-meson is produced either in $|K^0\rangle$ or in $|\bar{K}^0\rangle$ states. $|K^0\rangle$ produced can be regarded as a combination of $|K_1^0\rangle$ and $|K_2^0\rangle$ with definite phase relation.

What happens in weak decays? In weak interactions strangeness need not be conserved, so it can cause the transition of K^0 into its antiparticle $\bar{K}^0$. However, if *CP* is assumed in these decays, the states $|K_1^0\rangle$ and $|K_2^0\rangle$ decay with definite life times, say τ_1 and τ_2.

Let us study the nonleptonic decays of the neutral *K*-mesons. The decay final states of interest are:

(i) $\pi^0 + \pi^0$,
(ii) $\pi^+ + \pi^-$,
(iii) $\pi^0 + \pi^0 + \pi^0$,
(iv) $\pi^+ + \pi^- + \pi^0$.

Under *CP*, the states of two pions (i) and (ii) are necessarily even. So, if the *CP* invariance holds, then

$$K_1^0 \to 2\pi^0, \qquad \pi^+ + \pi^-.$$

However,

$$K_2^0 \not\to 2\pi^0, \qquad \pi^+ + \pi^-.$$

A $3\pi^0$ system is necessarily odd under *CP*, but the state of $\pi^+\pi^-\pi^0$ is odd or even depending upon the angular momentum:

$$CP = -1, \qquad \text{if } l = L = 1, 3, 5, \ldots$$

$$CP = +1, \qquad \text{if } l = L = 0, 2, 4, \ldots$$

l and L are orbital angular moments between the first two pions and between the centre of mass of the two pions and the third pion. The angular momentum is probably zero, since otherwise there would be a large angular momentum barrier which would depress the decay rate in the manner not observed. There is no evidence for $l \neq 0$. Thus the most likely value for *CP* for the decay mode (iv) is −1. Thus the dominant mode of K_1^0 is the 2π mode while the K_2^0 decays predominantly into 3π. On the basis of phase space we expect the rate of decay into three pions much smaller than that into two pions. The leptonic modes are also slower. So K_1^0 and K_2^0 have very different lifetimes. Therefore, as far as the decays are concerned, K_1^0 and K_2^0 rather than K^0 and $\bar{K}^0$ behave like different particles. Obviously, K_2^0 must live much longer. Experimentally,

$$\left.\begin{aligned} \tau_1 &= (0.893 \pm 0.003) \times 10^{-10}\ \text{s}, \\ \tau_2 &= (5.18 \pm 0.040) \times 10^{-8}\ \text{s}. \end{aligned}\right\} \tag{7.21}$$

The important decay modes of $K_S^0 \equiv K_1^0$ and $K_L^0 \equiv K_2^0$ are given in the Table 7.1.

7.4 Strangeness Oscillations

For a stable particle at rest, the time evolution is given by $\psi = e^{-iMt}$. For an unstable particle

Table 7.1 Neutral Kaon Decay Modes

Kaon	*Mean lifetime*	*Decay mode*	*Decay fraction (%)*
K_S^0	0.89×10^{-10}	$\pi^+\pi^-$	68.6
		$\pi^0\pi^0$	31.39
		$\mu^+\mu^-$	$< 3 \times 10^{-7}$
		e^+e^-	$< 1 \times 10^{-5}$
		$\pi^+\pi^-\gamma$	1.85×10^{-3}
		$\gamma\gamma$	2.4×10^{-6}
K_L^0	5.18×10^{-8} $m_1 - m_2 = 3.52 \times 10^{-12}$ MeV	$\pi^0\pi^0\pi^0$	21.7
		$\pi^+\pi^-\pi^0$	12.37
		$\pi \pm \mu^\mp \nu$	27.01
		$\pi^\pm e^\mp \nu$	38.6
		$\gamma\gamma$	0.57
		$\pi^+\pi^-$	0.204
		$\pi^0\pi^0$	0.204

at rest, this is modified into $\psi = e^{-i(M-i\lambda/2)t}$ (in fact, $|\psi|^2 = e^{-\lambda t} = e^{-t/\tau}$), with M and λ as real positive numbers. Let us see how a neutral K beam changes with time when initially we have only K^0, i.e.

$$|\psi(0)\rangle = |K^0\rangle = \frac{1}{\sqrt{2}}\{|K_1^0\rangle + |K_2^0\rangle\}. \tag{7.22}$$

At a later time t, we will have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\{|K_1^0\rangle e^{-\mathrm{i}m_1 t} e^{-\lambda_1 t/2} + |K_2^0\rangle e^{-\mathrm{i}m_2 t} e^{-\lambda_2 t/2}\}. \tag{7.23}$$

Here, λ_1 and λ_1 are the decay rates and m_1 and m_2 the masses of K_1^0 and K_2^0, respectively; m_1 and m_2 are slightly different because K_1^0 and K_2^0 have different weak interactions. From dimensional considerations the mass difference is of the order of $\lambda_1 = 1/\tau_1 = 7 \times 10^{-12}$ MeV, since both the self-energy effects responsible for the mass difference and the decay probability are of second order in the weak interaction. For the probability of observing K^0, we have

$$\begin{aligned} |\langle K^0|\psi(t)\rangle|^2 &= \frac{1}{4}[e^{-\lambda_1 t} + e^{-\lambda_2 t} + 2e^{-(\lambda_1+\lambda_2)t/2}\cos\{(m_1 - m_2)t\}] \\ &= \frac{1}{4}[e^{-\lambda_1 t} + 1 + 2e^{-\lambda_1 t/2}\cos\{(m_1 - m_2)t\}]. \end{aligned} \tag{7.24}$$

Since λ_2 is small, $e^{-\lambda_2 t} \approx 1$. Similarly for $\overline{K}^0$,

$$\begin{aligned} |\langle \overline{K}^0|\psi(t)\rangle|^2 &= \frac{1}{4}[e^{-\lambda_1 t} + e^{-\lambda_2 t} - 2e^{-(\lambda_1+\lambda_2)t/2}\cos\{(m_1 - m_2)t\}] \\ &= \frac{1}{4}[e^{-\lambda_1 t} + 1 - 2e^{-\lambda_1 t/2}\cos\{(m_1 - m_2)t\}] \end{aligned} \tag{7.25}$$

The intensity of the $\overline{K}^0$ component can be measured by studying the rate of the hyperon production. If the mass difference is large compared to λ_1, we expect a rapid oscillation between K^0 and $\overline{K}^0$. The intensity of K^0 starts at unity and as time passes this shows an oscillatory component which depends upon the mass difference. Similarly, the intensity of $\overline{K}^0$ initially rises from zero and then shows oscillations with the same frequency. After times much longer than τ_1 but still short compared to τ_2, only the $\overline{K}^0$ component remains and $K^0(t) = \overline{K}^0(t) = 1/4$. By fitting the observed data it is found that

$$(m_2 - m_1) = (3.521 \pm 0.014) \times 10^{-12}\ \text{MeV}. \tag{7.26}$$

This is remarkably close to the value of Γ_1, viz. 7.37 × 10^{-12} MeV.

7.5 *CP*-Violation

In 1964, J.H. Christiansen, J.W. Cronin, V.L. Fitch, and R. Turlay found 2π-meson decays of K_2^0 at a low rate, now measured as (0.094)%. This decay is a *CP* violating process. No other *CP* violation has been seen outside the *K*-meson system so far. This important discovery means that we cannot identify $|K_S^0\rangle$ with $|K_1^0\rangle$ and $|K_L^0\rangle$ with $|K_2^0\rangle$. Instead, if *CP* is violated accompanied by *T* violation, so that CPT is conserved, we have

$$|K_S^0\rangle = \frac{|K_1^0\rangle + \varepsilon|K_2^0\rangle}{\sqrt{(1+|\varepsilon|^2)}},$$

$$|K_L^0\rangle = \frac{|K_2^0\rangle + \varepsilon|K_1^0\rangle}{\sqrt{(1+|\varepsilon|^2)}},$$

where ε is a small complex number which measures the degree of *CP* violation due to kaon state mixing. We shall discuss now the *CP*-violation in *K*-decays in greater detail.

7.6 *CP*-Violation in *K*-Decays

7.6.1 Mixing of Neutral *K*-mesons

Let us consider neutral *K*-meson K^0 and its antiparticle $\overline{K}^0$. An arbitrary state at time *t* may be written as

$$|\psi(t)\rangle = A(t)|K^0\rangle + B(t)|\overline{K}^0\rangle. \tag{7.27}$$

In the basis of K^0 and $\overline{K}^0$, we can write this state simply as

$$\psi = \begin{pmatrix} A \\ B \end{pmatrix}. \tag{7.28}$$

The Hamiltonian has three parts, H_S, H_{EM} and H_{WK}, corresponding to the strong, electromagnetic and weak interactions. Thus,

$$H = H_S + H_{EM} + H_{WK} \tag{7.29}$$

The time development of ψ is governed by the time-dependent Schrödinger equation

$$H\psi = i\hbar \frac{d}{dt}\psi. \tag{7.30}$$

In the basis of K^0 and $\overline{K}^0$, H is a 2×2 matrix. Since the neutral kaons decay, H is not a Hermitian matrix and so can be written as

$$H = M + i\Gamma, \tag{7.31}$$

where M and Γ are 2×2 Hermitian matrices. The anti-Hermitian part $i\Gamma$ describes the exponential decay of the meson. M and Γ are called the mass and decay matrices respectively. K^0 and $\overline{K}^0$ are eigenstates of $H_S + H_{EM} = H_0$ with strangeness eigenvalue of +1 and −1 respectively. Since the strong and electromagnetic interactions conserve strangeness, we must have

$$\langle K^0 | H_0 | \overline{K}^0 \rangle = 0. \tag{7.32}$$

If $H = H_0$, i.e. if H_{WK} were zero, then K^0 and $\overline{K}^0$ would not decay and Γ would be zero. In that case,

$$H \to M = \begin{bmatrix} m_{K^0} & 0 \\ 0 & m_{\overline{K}^0} \end{bmatrix}, \tag{7.33}$$

where

$$m_{K^0} = \langle K^0 | H_0 | K^0 \rangle, \tag{7.33a}$$

$$m_{\overline{K}^0} = \langle \overline{K}^0 | H_0 | \overline{K}^0 \rangle. \tag{7.33b}$$

The CPT invariance of H_0 yields

$$m_{K^0} = m_{\overline{K}^0}. \tag{7.34}$$

Now let us introduce the weak interaction so that Γ is not zero. The matrix M will now have non-zero off diagonal elements. If we treat H_{WK} perturbatively, we have

$$\left.\begin{aligned} M_{11} &= m_{K^0} + \langle K^0 | H_{WK} | K^0 \rangle + \sum_n \frac{|\langle K^0 | H | n \rangle|^2}{m_{K^0} - E_n} + \ldots, \\ M_{22} &= m_{\overline{K}^0} + \langle \overline{K}^0 | H_{WK} | \overline{K}^0 \rangle + \sum_n \frac{|\langle \overline{K}^0 | H_{WK} | n \rangle|^2}{m_{K^0} - E_n} + \ldots, \\ M_{21} &= M_{12}^* = \langle \overline{K}^0 | H_{WK} | K^0 \rangle + \sum_n \frac{\langle \overline{K}^0 | H_{WK} | n \rangle \langle n | H_{WK} | K^0 \rangle}{m_{K^0} - E_n} + \ldots. \end{aligned}\right\} \tag{7.35}$$

If H_{WK} is CPT invariant, then

$$(\mathrm{CPT})^{-1} H_{WK} \mathrm{CPT} = H_{WK}, \tag{7.36}$$

$$\langle K^0|H_{WK}|n\rangle = \langle K^0|(\mathrm{CPT})^{-1}H_{WK}(\mathrm{CPT})|n\rangle = \langle \bar{K}^0|H_{WK}|n'\rangle^* = \langle n'|H_{WK}|K^0\rangle, \tag{7.37}$$

where the prime denotes reversed spins. However, since the spins are taken over the possible values of all states n, the CPT invariance gives

$$M_{11} = M_{22}. \tag{7.38}$$

Since the first order weak interactions obey the selection rules $|\Delta S| \leq 1$,

$$\langle K^0|H_{WK}|\bar{K}^0\rangle = 0. \tag{7.39}$$

Also, if H_{WK} is time reversal invariant,

$$M_{21} = M_{12}^* = M_{12}. \tag{7.40}$$

We shall find that the second order terms are important and must be retained. We can, however, ignore terms of higher orders.

7.6.2 Conservation of Probability

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle + 2\pi\sum_F \rho(F)|\langle F|H_{WK}|\psi(t)\rangle|^2 = 0, \tag{7.41}$$

$|F\rangle$ denotes a particular final state resulting from K^0 and $\bar{K}^0$ decay, and $\rho(F)$ is the density of final states. For an arbitrary state, we have from (7.30) and (7.31), the relations

$$\frac{d}{dt}|\psi(t)\rangle = -(\Gamma + iM)|\psi(t)\rangle, \tag{7.42}$$

$$\frac{d}{dt}\langle\psi(t)| = -\langle\psi(t)|(\Gamma - iM) \tag{7.43}$$

so that

$$\begin{aligned}\frac{d}{dt}\langle\psi|\psi\rangle &= \langle\psi|\frac{d}{dt}\psi\rangle + \langle\frac{d}{dt}\psi|\psi\rangle \\ &= -\langle\psi(t)|\Gamma + iM|\psi(t)\rangle - \langle\psi(t)|\Gamma - iM|\psi(t)\rangle \\ &= -2\langle\psi(t)|\Gamma|\psi(t)\rangle.\end{aligned} \tag{7.44}$$

Then from (7.41), we obtain

$$\langle\psi(t)|\Gamma|\psi(t)\rangle = \pi\sum_F \rho(F)|\langle F|H_{WK}|\psi(t)\rangle|^2. \tag{7.45}$$

Since the coefficients A and B are arbitrary, by comparing the coefficients of A^*A, B^*B and A^*B, we get

$$\Gamma_{11} = \langle K^0|\Gamma|K^0\rangle = \pi\sum_F \rho(F)|\langle F|H_{WK}|K^0\rangle|^2, \tag{7.46}$$

$$\Gamma_{22} = \langle\bar{K}^0|\Gamma|\bar{K}^0\rangle = \pi\Sigma\rho(F)|\langle F|H_{WK}|\bar{K}^0\rangle|^2, \tag{7.47}$$

$$\Gamma_{21}^* = \Gamma_{12} = \langle\bar{K}^0|\Gamma|K^0\rangle = \pi\sum_F\rho(F)\langle\bar{K}^0|H_{WK}|F\rangle\langle F|H_{WK}|K^0\rangle. \tag{7.48}$$

The matrix elements are thus expressed as sums over terms corresponding to different final states F. The CPT invariance gives

$$\begin{aligned}\langle F|H_{\mathrm{WK}}|K^0\rangle &= \langle F|(\mathrm{CPT})^{-1}H_{\mathrm{WK}}(\mathrm{CPT})|K^0\rangle \\ &= \langle \overline{K}^0|H_{\mathrm{WK}}|F'\rangle = \langle F'|H_{\mathrm{WK}}|\overline{K}^0\rangle^*, \qquad (7.49)\end{aligned}$$

where the prime denotes reversed spins. However, the sums are taken over all possible final states F and so the CPT invariance implies

$$\Gamma_{11} = \Gamma_{22}. \qquad (7.50)$$

If the T invariance holds for the weak interaction, then we can adjust phases so that $\langle F|H_{\mathrm{WK}}|K^0\rangle$ and $\langle F|H_{\mathrm{WK}}|\overline{K}^0\rangle$ are real. In this case it follows that, for T invariance,

$$\Gamma_{12} = \Gamma_{21} = \Gamma_{12}, \qquad (7.51)$$

$$M_{12} = M_{21} = M_{12}^*. \qquad (7.52)$$

7.6.3 Eigenvalues and Eigenvectors of $\Gamma + iM$

Let us write

$$\Gamma + iM = DI + i\mathbf{E}\cdot\sigma. \qquad (7.53)$$

I and σ are 2 × 2 identity and Pauli matrices. D and E are complex numbers. We can write

$$\begin{aligned}\Gamma + iM &= \begin{bmatrix} D + iE_3 & i(E_1 + iE_2) \\ i(E_1 - iE_2) & D - iE_3 \end{bmatrix} \\ &= \begin{bmatrix} D + iE\cos\theta & iE\sin\theta e^{-i\phi} \\ iE\sin\theta e^{i\phi} & D - iE\cos\theta \end{bmatrix}. \qquad (7.54)\end{aligned}$$

We have written $E\cos\theta$ for E_3 and $E\sin\theta\, e^{-i\phi}$ for $E_1 + iE_2$. To diagonalize, we put the determinant of the matrix

$$\begin{bmatrix} D + iE\cos\theta - \lambda & iE\sin\theta e^{-i\phi} \\ iE\sin\theta e^{i\phi} & D - iE\cos\theta - \lambda \end{bmatrix} \qquad (7.55)$$

equal to zero to obtain the eigenvalues

$$\lambda = D \mp iE. \qquad (7.56)$$

Let us call $\lambda_{\mathrm{S}} = D + iE$ and $\lambda_{\mathrm{L}} = D - iE$. Through

$$\lambda_{\mathrm{S}} \equiv \frac{1}{2}\gamma_{\mathrm{S}} + im_{\mathrm{S}}, \qquad \lambda_{\mathrm{L}} \equiv \frac{1}{2}\gamma_{\mathrm{L}} + im_{\mathrm{L}}. \qquad (7.57)$$

we define γ_{L}, γ_{S}, m_{L}, m_{S}, which are all real quantities corresponding to the eigenvalues of Γ and M for the states K_{L}^0 and K_{S}^0, respectively:

$$(\Gamma + iM)|\,K_S^0\rangle = \lambda_S|\,K_S^0\rangle = (D + iE)|\,K_S^0\rangle, \tag{7.58}$$

$$(\Gamma + iM)|\,K_L^0\rangle = \lambda_L|\,K_L^0\rangle = (D - iE)|\,K_L^0\rangle. \tag{7.59}$$

The states $|\,K_S^0\rangle$ and $|\,K_L^0\rangle$ have definite lifetimes $\tau_S = 1/\gamma_S$ and $\tau_L = 1/\gamma_L$. The time dependence of these states is governed by

$$|K_S^0(t)\rangle = |K_S^0\rangle\, e^{-im_S t}\, e^{-\gamma_S t/2} \tag{7.60a}$$

$$|K_L^0(t)\rangle = |K_L^0\rangle\, e^{-im_L t}\, e^{-\gamma_L t/2}. \tag{7.60b}$$

In general, the *CP* violation in neutral kaon decays can be parametrized as follows:

$$|K_S^0\rangle = [(1 + \varepsilon_1)|K^0\rangle - (1 - \varepsilon_1)|\overline{K}^0\rangle]\,[2(1 + |\varepsilon_1|^2)]^{-1/2}, \tag{7.61a}$$

$$|K_L^0\rangle = [(1 + \varepsilon_2)|K^0\rangle + (1 - \varepsilon_2)|\overline{K}^0\rangle]\,[2(1 + |\varepsilon_2|^2)]^{-1/2}, \tag{7.61b}$$

where ε_1, ε_2 are complex numbers. If *CP* violation is accompanied by *T* violation so that CPT is conserved, then $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

7.6.4 Transition Amplitudes for $K_S^0 \to 2\pi$ and $K_L^0 \to 2\pi$

Let us now calculate the decays of K_S^0 and K_L^0 into two pions. The final-state pions are bosons, so the total wave function must be symmetric with respect to particle interchange. The isospin of a pion is one and so the final state should have isospin 0 or 2 with $I_3 = 0$. There are four amplitudes:

$$\langle \pi\pi, I = 0\,|H_{WK}|\,K_S^0\rangle, \qquad \langle \pi\pi, I = 2\,|H_{WK}|\,K_S^0\rangle, \tag{7.62a}$$

$$\langle \pi\pi, I = 0\,|H_{WK}|\,K_L^0\rangle, \qquad \langle \pi\pi, I = 2\,|H_{WK}|\,K_L^0\rangle. \tag{7.62b}$$

Using Clebsch-Gordan coefficients, we can write

$$|\,\pi^+\pi^-\rangle = \sqrt{\frac{1}{3}}\,|\,\pi\pi, I = 2\rangle + \sqrt{\frac{2}{3}}\,|\,\pi\pi, I = 0\rangle, \tag{7.63a}$$

$$|\,\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}}\,|\,\pi\pi, I = 2\rangle - \sqrt{\frac{1}{3}}\,|\,\pi\pi, I = 0\rangle \tag{7.63b}$$

The pions undergo phase shifts δ_0 and δ_2 respectively in the two final states. Let us define the decay amplitudes as

$$\langle \pi\pi, I = 0\,|H_{WK}|K^0\rangle = A_0 e^{i\delta_0}, \tag{7.64a}$$

$$\langle \pi\pi, I = 2\,|H_{WK}|K^0\rangle = A_2 e^{i\delta_2}. \tag{7.64b}$$

The amplitudes for K^0 can be deduced from symmetry. Under CPT,

$$|K^0\rangle \to \langle \overline{K}^0\,|, \tag{7.65}$$

and so if the weak interaction is invariant under CPT, we have

$$\langle \pi\pi, I = 0 | H_{\mathrm{WK}} | \overline{K}^0 \rangle = A_0^* e^{i\delta_0}, \tag{7.66a}$$

$$\langle \pi\pi, I = 2 | H_{\mathrm{WK}} | \overline{K}^0 \rangle = A_2^* e^{i\delta_2}. \tag{7.66b}$$

An arbitrary overall phase may be eliminated by taking A_0 real. Experimentally, $\left| \frac{A_2}{A_0} \right| \sim \frac{1}{20}$.

$$\langle \pi^+\pi^- | H_{\mathrm{WK}} | K_{\mathrm{L}}^0 \rangle = \{6(1 + |\varepsilon|^2)\}^{-1/2}, \tag{7.67a}$$

$$[(A_2 - A_2^*)e^{i\delta_2} + \varepsilon\{(A_2 + A_2^*)e^{i\delta_2} + 2\sqrt{2}\, A_0\, e^{i\delta_2}\}], \tag{7.67b}$$

$$\langle \pi^+\pi^- | H_{\mathrm{WK}} | K_{\mathrm{S}}^0 \rangle = \{6(1 + |\varepsilon|^2)\}^{-1/2},$$

$$[(A_2 + A_2^*)e^{i\delta_2} + 2\sqrt{2}\, A_0\, e^{i\delta_2} + \varepsilon(A_2 + A_2^*)e^{i\delta_2}], \tag{7.68a}$$

$$\langle \pi^0\pi^0 | H_{\mathrm{WK}} | K_{\mathrm{L}}^0 \rangle = \sqrt{\frac{2}{3}}\, \{2(1 + |\varepsilon|^2)\}^{-1/2},$$

$$[(A_2 - A_2^*)e^{i\delta_2} + \varepsilon\{(A_2 + A_2^*)e^{i\delta_2} - \sqrt{2}\, A_0\, e^{i\delta_2}\}], \tag{7.68b}$$

$$\langle \pi^+\pi^- | H_{\mathrm{WK}} | K_{\mathrm{S}}^0 \rangle = \sqrt{\frac{2}{3}}\, \{2(1 + |\varepsilon|^2)\}^{-1/2},$$

$$[(A_2 - A_2^*)e^{i\delta_2} - \sqrt{2}\, A_0\, e^{i\delta_2} + \varepsilon(A_2 + A_2^*)e^{i\delta_2}]. \tag{7.69}$$

So if we neglect second order terms, we obtain the ratio

$$\begin{aligned}
\eta^{+-} &= \frac{\langle \pi^+\pi^- | H_{\mathrm{WK}} | K_{\mathrm{L}}^0 \rangle}{\langle \pi^+\pi^- | H_{\mathrm{WK}} | K_{\mathrm{S}}^0 \rangle} \\
&= \frac{2i\,\mathrm{Im}\,(A_2)\, e^{i\delta_2} + 2\,\varepsilon\,\mathrm{Re}(A_2)\, e^{i\delta_2} + 2\sqrt{2}\,\varepsilon A_0\, e^{i\delta_0}}{2\,\mathrm{Re}\,(A_2)\, e^{i\delta_2} + 2i\,\varepsilon\,\mathrm{Im}(A_2)\, e^{i\delta_2} + 2\sqrt{2}\, A_0\, e^{i\delta_0}} \\
&= \frac{\varepsilon + \frac{1}{\sqrt{2}}\,(i\,\mathrm{Im}\,(A_2)/A_2)\, e^{i(\delta_2 - \delta_0)}}{1 + (i\,\mathrm{Re}\,(A_2)/\sqrt{2}\,A_0)\, e^{i(\delta_2 - \delta_0)} + \frac{1}{\sqrt{2}}\,\varepsilon(\mathrm{Im}\,(A_2)/A_0)\, e^{i(\delta_2 - \delta_0)}} \\
&= \varepsilon + \varepsilon'
\end{aligned} \tag{7.70}$$

and

$$\begin{aligned}
\eta^{00} &= \frac{\langle \pi^0\pi^0 | H_{\mathrm{WK}} | K_{\mathrm{L}}^0 \rangle}{\langle \pi^0\pi^0 | H_{\mathrm{WK}} | K_{\mathrm{S}}^0 \rangle} \\
&= \frac{2i\,\mathrm{Im}\,(A_2)\, e^{i\delta_2} + 2\,\varepsilon\,\mathrm{Re}(A_2)\, e^{i\delta_2} - \sqrt{2}\,\varepsilon A_0\, e^{i\delta_0}}{2\,\mathrm{Re}\,(A_2)\, e^{i\delta_2} + 2i\,\varepsilon\,\mathrm{Im}\,(A_2)\, e^{i\delta_2} - \sqrt{2}\, A_0\, e^{i\delta_0}} \\
&= \varepsilon - 2\varepsilon',
\end{aligned} \tag{7.71}$$

where

$$\varepsilon' = \frac{1}{\sqrt{2}}\, i\,\mathrm{Im}\left(\frac{A_2}{A_0}\right) e^{i(\delta_2 - \delta_0)}. \tag{7.72}$$

Observation of nonzero value of ε' would imply the existence of direct *CP* violation, i.e. *CP* violation without kaon state mixing.

The current values of the *CP* violation parameters (Particle Data Group) are:

$$|\eta^{00}| = (2.253 \pm 0.024) \times 10^{-3}, \qquad |\eta^{+-}| = (2.268 \pm 0.023) \times 10^{-3}$$

$$\phi^{00} = (44.6 \pm 2.0)^0, \qquad \phi^{+-} = (44.6 \pm 1.2)^0, \qquad \delta = (0.327 \pm 0.012)\%$$

where $\eta = |\eta| e^{\phi}$.

Evidence for direct CP violation was first observed in an experiment at CERN in 1988 corresponding to $\varepsilon'/\varepsilon = (3.3 \pm 1.1) \times 10^{-3}$, but the measurements at Fermilab are consistent with zero.

PROBLEMS

1. For the weak decay of a $1/2^+$-baryon a with momentum p into another $1/2^+$-baryon b of momentum p' and a 0^--meson c of momentum q, the matrix element can be written as

$$M = \bar{u}(p)[A + B\gamma_5]u(p').$$

Using the formula

$$\Gamma = \frac{1}{32\pi^2 M_a^2} |\mathbf{q}| \sum_{\text{spin}} |M|^2 \, d\Omega$$

for the decay rate, and the formula

$$\sum_{\text{spin}} u(p)\bar{u}(p) = \gamma \cdot p + m$$

for the sum of polarizations, show that

$$\Gamma = \frac{1}{4\pi} \frac{|\mathbf{q}|\,(E_b + M_b)}{M_a} \left[|A|^2 + \frac{E_b - M_b}{E_b + M_b} |B|^2\right].$$

2. The branching ratios for the $\Lambda \to p\pi^-$ and $\Lambda \to n\pi^0$ are 64.2% and 35.8%, respectively. What would you expect if the nonleptonic Hamiltonian obeys $\Delta I = 1/2$ rule?

3. Find the ratio of the lifetimes Ξ^- and Ξ^0.

4. Use the $\Delta I = 1/2$ rule to explain the following decay rates of K-mesons:

$$\Gamma(K^+ \to \pi^+\pi^0) =, \qquad \Gamma(K_S^0 \to \pi^+\pi^-) =, \qquad \Gamma(K_S^0 \to \pi^0\pi^0) =$$

Solutions

Weak Interactions

Let us consider the decay

$$a(p) \to b(p') + c(q)$$

of a $1/2^+$-baryon a with four momentum p into another $1/2^+$-baryon b of four momentum p' and a 0^--meson of four momentum q. For weak interactions, the parity is not conserved, the matrix element for the decay will, therefore, be a sum of a scalar term and a pseudoscalar term. Then using Lorentz covariance, we can write the matrix element as

$$M = \bar{u}(p')[A + B\gamma_5]u(p),$$

where A is the parity violating (s-wave) amplitude, whereas B is the parity conserving (p-wave) amplitude. Squaring the m.e. and averaging over the initial spins and summing over the final spins, we get

$$\frac{1}{2}\sum_{\text{spin}} M^\dagger M = \frac{1}{2}\,\text{Trace}\,[(\gamma\cdot p + m_a)(A^* - B^*\gamma_5)(\gamma\cdot p' + m_b)(A + B\gamma_5)]$$

$$= 2(p\cdot p' + m_b m_a)\left[|A|^2 + \frac{(p\cdot p' - m_a m_b)}{(p\cdot p' + m_a m_b)}|B|^2\right]$$

$$= 2m_a(E_b + m_b)\left[|A|^2 + \frac{E_b - m_b}{E_b + m_b}|B|^2\right]$$

so that

$$\Gamma = \frac{1}{4\pi}\frac{|\mathbf{q}|}{m_a}(E_b + m_b)\left[|A|^2 + \frac{E_b - m_b}{E_b + m_b}|B|^2\right],$$

where

$$|\mathbf{q}| = \frac{1}{2m_a}[\{m_a^2 - (m_b + m_c)^2\}\{m_a^2 - (m_b - m_c)^2\}]^{1/2},$$

$$E_b + m_b = \frac{(m_a + m_b)^2 - m_c^2}{2m_a},$$

$$E_b - m_b = \frac{(m_a - m_b)^2 - m_c^2}{2m_a}.$$

We can also write

$$\Gamma = C_1\,[|A|^2 + C_2|B|^2],$$

where

$$C_1 = \frac{1}{8\pi}|\mathbf{q}|\frac{[(m_a + m_b)^2 - m_c^2]}{m_a^2}, \qquad C_2 = \frac{(m_a - m_b)^2 - m_c^2}{(m_a + m_b)^2 - m_c^2}.$$

8

GAUGE INVARIANCE

8.1 Introduction

The electromagnetic field is coupled to conserved currents of charged matter fields. This property is derived from a "gauge principle", i.e. the electromagnetic coupling arises when we extend "global gauge invariance" to "local gauge invariance".

8.2 Abelian Transformations: $U(1)$ Gauge Symmetry

To illustrate the gauge principle, we first consider a complex scalar field $\phi(x)$. The Lagrangian density of a complex Klein-Gordon field in the absence of electromagnetic coupling is

$$\mathcal{L}_0 = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{8.1}$$

The Euler-Lagrange equation gives

$$(\partial_\mu\partial^\mu + m^2)\phi = 0. \tag{8.2}$$

The Lagrangian density (8.1) is invariant under the transformation

$$\phi(x) \to e^{iq\lambda}\phi(x), \qquad \phi^*(x) \to e^{-iq\lambda}\phi^*(x) \tag{8.3}$$

because

$$\partial^\mu\phi(x) \to e^{iq\lambda}\partial^\mu\phi(x), \qquad \partial_\mu\phi^*(x) \to e^{-iq\lambda}\partial_\mu\phi^*(x), \tag{8.4}$$

In (8.3) and (8.4), q is a constant and represents the charge of the particle corresponding to the field ϕ. λ is an arbitrary constant. Since λ does not depend upon space-time point, this transformation is called global gauge transformation and the theory is said to be global gauge invariant under the group $U(1)$. Noether's theorem tells us that there is a conserved current j^μ given by

$$j^\mu = \phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi \tag{8.5}$$

satisfying

$$\partial_\mu j^\mu = 0 \tag{8.6}$$

or

$$\frac{\partial j^0}{\partial t} + \nabla\cdot\mathbf{J} = 0,$$

and yielding

$$\frac{d}{dt}\int d^3x\, j^0 = -\int d^3x\, \nabla\cdot\mathbf{J} = \int ds\, \mathbf{n}\cdot\mathbf{J} = 0, \tag{8.7}$$

where $\int d^3xj^0 = Q$ is called a charge of the field in general and is the generator of the gauge transformation. If Q is the electric charge operator, we call the transformation as the gauge transformation in the charge space. These transformations form a group of continuous transformations with one parameter only and so are termed $U(1)$. In fact, all additive quantum numbers such as the lepton number, the baryon number and the strangeness, etc. will generate gauge transformations corresponding to the group $U(1)$.

8.2.1 Local Gauge Transformations

We would like to see what would happen if we allow the parameter λ to depend upon the space-time coordinates. Thus,

$$\phi(x) \to e^{iq\lambda(x)}\phi(x), \qquad \phi^*(x) \to e^{-iq\lambda(x)}\phi^*(x). \tag{8.8}$$

The important point to note here is that the Lagrangian density (8.1) is not invariant under this more general transformation because the derivative acts on phase also and so transformation becomes

$$\partial^\mu\phi(x) \to e^{iq\lambda(x)}\partial^\mu\phi(x) + i(q\partial^\mu\lambda(x))\, e^{iq\lambda(x)}\phi(x). \tag{8.9}$$

In other words, the derivative of the field no longer transforms like the field itself and consequently, the Lagrangian (8.1) is no longer invariant under (8.8). To restore the invariance, we introduce a vector field A^μ and a new derivative D^μ, called the covariant derivative, which is again first order differential operator, but has the property that it transforms under (8.8) like the field itself:

$$D^\mu\phi(x) \to e^{iq\lambda(x)}D^\mu\phi(x). \tag{8.10}$$

The field $A^\mu(x)$ is called the gauge field and, by definition, transforms like

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\lambda(x). \tag{8.11}$$

We then define

$$D^\mu \equiv \partial^\mu + iqA^\mu, \tag{8.12}$$

and it is easy to see that $D^\mu\phi(x)$ indeed transforms under (8.8) and (8.11) as does (8.10).

The invariance under the local gauge transformation is then restored by replacing ∂^μ by D^μ in the Lagrangian density $\mathcal{L}_0$.

$$\begin{aligned} L_0 \to L_I &\equiv D_\mu\phi^*D^\mu\phi - m^2\phi^*\phi = (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi \\ &= \partial_\mu\phi^*\partial^\mu\phi - ieA_\mu\{\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\} + e^2A_\mu A^\mu\phi^*\phi - m^2\phi^*\phi. \end{aligned} \tag{8.13}$$

This Lagrangian density is invariant under (8.8) and (8.11) and it contains the gauge field A_μ. We will have to add to (8.13) its kinetic energy, the derivative terms. This term must be gauge invariant also. If we define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{8.14}$$

we see that $F^{\mu\nu}$ is invariant under the transformation (8.11). We are thus led to the final Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_I = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi. \tag{8.15}$$

We have achieved the invariance under the local gauge transformation by introducing a vector field A_μ. If this vector field A_μ is the electromagnetic vector potential and $F^{\mu\nu}$ is the electromagnetic field tensor, the Lagrangian in (8.15) then describes a charged Klein-Gordon field interacting with the electromagnetic field. Let us note that (8.15) does not contain a mass term $\frac{1}{2}M^2A_\mu A^\mu$, since such a term would violate the invariance under (8.11). In other words, gauge invariance forces the vector boson to be massless. This is correct for the photon which is massless.

8.3 Electrodynamics of a Dirac Field

Let us first consider for the free Lagrangian density of a Dirac field

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi \tag{8.16}$$

and the global transformations

$$\psi \to \psi' = e^{iq\lambda}\psi, \tag{8.17}$$

$$\bar{\psi} \to \bar{\psi}' = e^{-iq\lambda}\bar{\psi}. \tag{8.18}$$

Here q is the charge of the particle corresponding to the Dirac field. In the quantum field theory it will be replaced by the charge operator Q. Obviously, the Lagrangian density is invariant under this transformation and so we have, *from Noether's theorem*, a conserved current given by

$$J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{8.19}$$

This leads to the conservation of charge for the Dirac field.

8.3.1 Local Gauge Transformations

We would now like to see what would happen if we allow the parameter λ to depend upon the space-time coordinates. Let us now take

$$\psi \to \psi' = e^{iq\lambda(x)}\psi, \tag{8.20}$$

$$\bar{\psi} \to \bar{\psi}' = e^{-iq\lambda(x)}\bar{\psi}. \tag{8.21}$$

These are known as gauge transformations of the first kind. The free Lagrangian density (8.16) is no longer invariant as the derivative not only acts on the field ψ, it operates on $\lambda(x)$ also. Then we must introduce a massless vector field A_μ and start instead, with the Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi, \tag{8.22}$$

where

$$D_\mu \equiv \partial_\mu + iqA_\mu.$$

Then

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \tag{8.23}$$

with $\mathcal{L}_0$ = the free Lagrangian density of the Dirac field

$$\mathcal{L}_I = -q\,\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu \tag{8.24}$$

which represents the interaction of the Dirac field and the electromagnetic field with strength q. It is easy to see that the Lagrangian density given by (8.22) is invariant under the combined transformations of the first kind (8.17) and (8.18) and of the second kind

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\lambda(x). \tag{8.25}$$

Finally, as in (8.15), to the Lagrangian of (8.22), we add the kinetic energy of the electromagnetic field.

The requirement of the local gauge invariance (which may be taken as the principle of local gauge invariance) introduces a massless gauge boson field A_μ. As a result, we are led to the interaction of fields.

In summary, the unobservability of the local phases of the field (due to the local gauge invariance) has led to the existence of interactions, mediated by massless spin one (gauge) bosons.[1] In general, suppose we want the theory to be invariant under a transformation where the particle states change as

$$\psi' = U\psi \tag{8.26}$$

for some U. We want to define

$$D^\mu = \partial^\mu + igA^\mu, \tag{8.27}$$

where A^μ represents the interacting field which needs to be introduced to keep the theory invariant, but now we do not know how A^μ itself transforms. We also want

$$D^{\mu\prime}\psi' = U(D^\mu\psi), \tag{8.28}$$

and we can write this as

$$(\partial^\mu + igA^{\mu\prime})\,U\psi = U(\partial^\mu + igA^\mu)\psi. \tag{8.29}$$

This can be solved for $A^{\mu\prime}$ to give

$$igA^{\mu\prime}U\psi = [-\,\partial^\mu(U\psi) + U\partial^\mu\psi] + igUA^\mu\psi = -\,(\partial^\mu U)\psi + igUA^\mu\psi,$$

$$A^{\mu\prime}U\psi = \frac{i}{g}(\partial^\mu U)\psi + UA^\mu\psi.$$

---

[1] The masslessness of the mediators, necessitated by the gauge principle, is alright for QED, but catastrophic for weak interactions. We know from experiments that the weak mediators are massive.

Now since ψ is arbitrary, we can drop it to get

$$A^{\mu\prime}U = \frac{i}{g}(\partial^{\mu}U) + UA^{\mu}. \tag{8.30}$$

Multiply (8.30) from the right by U^{-1} and obtain

$$A^{\mu\prime} = UA^{\mu}U^{-1} + \frac{i}{g}(\partial^{\mu}U)U^{-1} = UA^{\mu}U^{-1} - \frac{i}{g}U\partial_{\mu}U^{-1}. \tag{8.31}$$

since $\partial^{\mu}(UU^{-1}) = 0$ and so $(\partial^{\mu}U)\, U^{-1} = -\, U(\partial^{\mu}U^{-1})$. This then gives the general transformation rule for the gauge field.

8.4 Spontaneous Symmetry Breaking

It is possible that the vacuum or the ground state of a physical system has less symmetry than the Lagrangian density. This possibility is called Spontaneous Symmetry Breaking (SSB). The phenomenon of SSB is known to occur in several branches of physics. Usually, systems exhibiting SSB have an infinite number of degrees of freedom. Since field theories are intrinsically theories with infinite number of degrees of freedom, it might be that the theory of fundamental constituents of matter is a spontaneously broken one. In fact, SSB plays a pivotal role in the standard model in the unification of weak and electromagnetic interactions.

8.4.1 Spontaneous Breaking of Global Symmetry: Goldstone model

Let us consider a system described by a Lagrangian density L which is invariant under a continuous group G of transformations. We examine the ground state of the system. If the system possesses a unique ground state, invariant under G, we have a situation where the symmetry properties of L and the vacuum (ground state) are the same. Such a theory is a normal one. On the other hand, it might happen that the system has several ground states which transform into each other under the transformations in G. Then, if for some reason, one of the ground states is singled out as the physical ground state of the system (the others being unphysical), the symmetry is lost and the theory is said to be spontaneously broken. The simplest example is the complex scalar field theory with $\mu^2 < 0$. Let $\phi(x)$ be a complex scalar field whose dynamics is described by the Lagrangian density L given by

$$L = \partial_{\mu}\phi^{*}\partial^{\mu}\phi - \mu^2\phi^{*}\phi - \lambda(\phi^{*}\phi)^2. \tag{8.32}$$

L is invariant under the group $U(1)$ of global transformations (8.3). The first term is the kinetic energy and the potential energy is given by

$$V = \mu^2\phi^{*}\phi + \lambda(\phi^{*}\phi)^2.$$

So the energy density $\mathcal{H}$ is

$$\mathcal{H} = \partial_{\mu}\phi^{*}\partial^{\mu}\phi + \mu^2\phi^{*}\phi + \lambda(\phi^{*}\phi)^2 \tag{8.33}$$

which is a sum of kinetic and potential energies of $\phi(x)$ and $\phi^{*}(x)$ fields. The complex field is obtained as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2),$$

where ϕ_1, ϕ_2 are real fields. $\lambda > 0$ as V has a minimum only if $\lambda > 0$, because otherwise $V(|\phi|) \to -\infty$ as $|\phi| \to \infty$, i.e. the Hamiltonian has no minimum and hence no ground state. In this case the position of the minimum depends on the sign of μ^2. Suppose that $\mu^2 \neq 0$. Then there are two possibilities:

(i) $$\mu^2 > 0$$

when we have a self-interacting charged scalar field.

(ii) $$\mu^2 < 0$$

where we have an example of spontaneously broken theory. Note that the kinetic energy $\partial_\mu \phi^* \partial^\mu \phi$ is non-negative and so is zero only if ϕ is a constant.

For possibility (i), $\mathcal{H} = 0$ if $\phi = 0$ giving $\phi_1 = \phi_2 = 0$ for the vacuum. Thus there is a unique vacuum at the origin in the $\phi_1 - \phi_2$ plane. $\mathcal{L}$ *is invariant under the* $U(1)$ *group of phase transformations* (8.3). The unique vacuum is also invariant under (8.3).

However, if $\mu^2 < 0$, as in possibility (ii), then there are nonzero values of $\phi(x)$ for which $\mathcal{H} = 0$. Thus there is a nonzero field configuration with lowest energy. The vacuum solution is found by solving the equation

$$\frac{\partial V}{\partial \phi} = 0 \tag{8.34}$$

or

$$\mu^2 \phi^* + 2\lambda(\phi^*\phi)\phi^* = 0. \tag{8.35}$$

So

$$\phi^*\phi = -\frac{\mu^2}{2\lambda} = \frac{1}{2}(\phi_1^2 + \phi_2^2) \tag{8.36}$$

for $\phi \neq 0$. Therefore,

$$(\phi_1^2 + \phi_2^2) = \frac{-\mu^2}{\lambda} = v^2 \text{ (say).} \tag{8.37}$$

Thus the system has an infinite number of ground states, which lie on a circle in the $\phi_1 - \phi_2$ plane. Under the action of the transformation (8.8), the ground states are transformed into each other. Suppose now we choose one of the ground states, e.g. $\phi_0 \equiv (\phi_1 = v, \phi_2 = 0)$ as the physical ground state, then the gauge symmetry is spontaneously broken because the state ϕ_0 is not invariant under the transformations. The *Lagrangian density* is still invariant. To see the structure of the theory, put

$$\phi = \frac{1}{\sqrt{2}}[v + \phi_1'(x) + i\phi_2'(x)]. \tag{8.38}$$

Then

$$\phi^*\phi = \frac{1}{2}[(v + \phi_1')^2 + \phi_2'^2] = \frac{1}{2}[v^2 + 2v\phi_1' + \phi_1'^2 + \phi_2'^2]. \tag{8.39}$$

Substituting (8.39) in (8.32), we get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1')(\partial^\mu \phi_1') + \frac{1}{2}(\partial_\mu \phi_2')(\partial^\mu \phi_2') - \frac{1}{2}\mu^2[v^2 + 2v\phi_1' + (\phi_1'^2 + \phi_2'^2)]$$

$$- \frac{1}{4}\lambda[v^4 + 4v^2\phi_1'^2 + (\phi_1'^2 + \phi_2'^2)^2 + 4v^3\phi_1' + 2v^2(\phi_1'^2 + \phi_2'^2)$$

$$+ 4v\phi_1'(\phi_1'^2 + \phi_2'^2)]. \tag{8.40}$$

With the help of $\lambda v^2 = -\mu^2$, (8.40) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1')(\partial^\mu \phi_1') + \frac{1}{2}(\partial_\mu \phi_2')(\partial^\mu \phi_2') + \frac{1}{4}\frac{\mu^4}{\lambda} + \mu^2\phi_1'^2$$

$$- \lambda v\phi_1'(\phi_1'^2 + \phi_2'^2) - \frac{1}{4}\lambda(\phi_1'^2 + \phi_2'^2)^2. \tag{8.41}$$

This Lagrangian density has the following features:

- The fields ϕ_1' and ϕ_2' have standard kinetic energy terms.
- The term $\mu^2\phi_1'^2$ can be interpreted as the mass term for the field ϕ_1'. The field $\phi_1'(x)$ thus describes a particle with mass μ.
- The field $\phi_2'(x)$ has no mass term and so describes a massless particle.
- The gauge symmetry is hidden in the Lagrangian density. Nevertheless, it has the gauge symmetry as can be seen by working backward.

This model is an example of a very general result known as Goldstone theorem. This theorem states that in any field theory, there is a zero mass spinless particle for each independent global continuous symmetry of the Lagrangian which is spontaneously broken. The zero mass particle is called a Goldstone boson.[1]

8.4.2 Spontaneous Breaking of a Local Symmetry

We start with the same Lagrangian density as in Section 8.4.1 and demand invariance under local phase transformation. From Section 8.4.1, we know that we must introduce a massless vector field and replace the derivative ∂_μ by the covariant derivative D_μ. A fully phase invariant generalization of (8.28) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{8.42}$$

We will see how the gauge field A_μ becomes massive through spontaneous symmetry breaking. This is the key to the understanding of electroweak theory. Much of the calculations from (8.33) to (8.41) can be translated to the Lagrangian density of (8.41). We choose for the ground state ϕ_0 with $\phi_1 = v$ and $\phi_2 = 0$ and using for ϕ the expression given in (8.38), we obtain

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[1] This general result does not apply to the local symmetry.

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\phi_1')(\partial^\mu\phi_1') + \frac{1}{2}(\partial_\mu\phi_2')(\partial^\mu\phi_2') - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2[v^2 + 2v\phi_1' \\ & + (\phi_1'^2 + \phi_2'^2)] - \frac{1}{4}\lambda[v^4 + 4v^2\phi_1'^2 + (\phi_1'^2 + \phi_2'^2)^2 + 4v^3\phi_1' + 2v^2(\phi_1'^2 + \phi_2'^2) \\ & + 4v\phi_1'(\phi_1'^2 + \phi_2'^2)] + \frac{e^2v^2}{2}A_\mu A^\mu - evA^\mu(\partial_\mu\phi_2') \\ & + eA^\mu(\partial_\mu\phi_1')\phi_2' + eA^\mu(\partial_\mu\phi_2')\phi_1' + e^2A^\mu A_\mu\phi_1' + \frac{e^2}{2}A^\mu A_\mu, \end{aligned} \tag{8.43}$$

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\phi_1')(\partial^\mu\phi_1') + \frac{1}{2}(\partial_\mu\phi_2')(\partial^\mu\phi_2') - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\frac{\mu^4}{\lambda} + \mu^2\phi_1'^2 \\ & - \lambda v\phi_1'(\phi_1'^2 + \phi_2'^2) - \frac{1}{4}\lambda(\phi_1'^2 + \phi_2'^2)^2 + \frac{e^2v^2}{2}A_\mu A^\mu - evA^\mu(\partial_\mu\phi_2') \\ & + eA^\mu(\partial_\mu\phi_1')\phi_2' + eA^\mu(\partial_\mu\phi_2')\phi_1' + e^2A^\mu A_\mu\phi_1' + \frac{e^2}{2}A^\mu A_\mu. \end{aligned} \tag{8.44}$$

This Lagrangian density apparently describes the interaction of a massive vector particle A_μ and two scalars, one massive (ϕ) and one massless (ϕ). However, we can immediately see that something is amiss in this counting. Let us count, for example, the number of degrees of freedom before and after the translation (8.40). For Lagrangian before the transformation, the field and degrees of freedom are as tabulated:

Field	*No. of degrees of freedom*
One massless vector field	2
Two scalar fields	2
Total	4

For Lagrangian after the transformation:

Field	*No. of degrees of freedom*
One massive vector field	3
Two scalar fields	2
Total	5

This tells us that we should be careful before interpreting the particle spectrum. Since physical degrees of freedom cannot be created by a simple change of variables, we conclude that the Lagrangian density must contain superfluous fields. We can make a transformation through which the unphysical field disappears. Let

$$\left.\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}[v + \rho(x)]\,e^{i\zeta(x)}, \\ A_\mu(x) &= B_\mu + \frac{1}{ev}\partial_\mu\zeta. \end{aligned}\right\} \tag{8.45}$$

The Lagrangian density transforms to

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu}B^{\mu\nu} + \frac{e^2v^2}{2} B_\mu^2 + \frac{1}{2} (\partial_\mu\rho)^2 - \frac{1}{2}(2\lambda v^2)\rho^2 - \frac{\lambda}{4}\rho^4 + \frac{1}{2}e^2B_\mu^2(2v\rho + \rho^2). \quad (8.46)$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

The field ζ has been gauged away. The Lagrangian density (8.46) describes, on quantization, two massive particles, a vector (B_μ) and a scalar (ρ). We can say that the vector meson has eaten the Goldstone boson and become heavy. This trick was discovered by P. Higgs and is called *Higgs' phenomenon*. The particle corresponding to $\rho(x)$ is normally referred to as a Higgs' particle. Let us note here that we have obtained three different Lagrangians describing the same physical system. $\mathcal{L}$ in (8.32) is invariant under the usual gauge transformations, but it has a negative mass for the field and so it is not suitable for quantization. $\mathcal{L}$ in (8.41) is gauge invariant but the transformation laws are more complicated. It can be quantized in a space containing unphysical degrees of freedom. This, by itself, is not a great obstacle and it happens frequently. For example, ordinary quantum electrodynamics is quantized in a space involving unphysical photons (longitudinal as well as scalar photons). In fact, it is this Lagrangian which, in a suitable gauge, is used for proving renormalizability, and for practical calculations. Finally, $\mathcal{L}$ in (8.43) is not invariant under any kind of gauge transformation, but it exhibits clearly the particle spectrum of the theory. It contains only the physical particles and they are all massive. Actually, $\mathcal{L}$ in (8.43) can easily be obtained from $\mathcal{L}$ in (8.32) by specifying the gauge of the latter. The $\mathcal{L}$ is non-renormalizable, by power counting, but, since it is gauge invariant equivalent to $\mathcal{L}$, it can still be used for practical calculations.

This analysis can be repeated for non-Abelian gauge theories with identical results as we indicate in later sections.

The conclusions can be summarized as follows:

In a spontaneously broken gauge symmetry the gauge vector bosons acquire a mass and would be Goldstone bosons decouple and disappear. Their degrees of freedom are used in order to give mass to the vector bosons. This phenomenon is known as *Higgs' mechanism*.

It is widely believed that all masses are generated via the Higgs' mechanism. It is appealing to imagine that a massless world built on the principle of local gauge invariance suddenly undergoes a spontaneous symmetry breaking whereby the fields acquire masses. Ever since the discovery by G. t'Hooft that SSB does not spoil the renormalizability, spontaneously broken gauge theories have been used to describe the interactions of the fundamental constituents of matter.

8.5 Non-Abelian Transformations

Let $\phi^i(x)$, $i = 1, 2, \dots n$ be a set of fields and $\mathcal{L}_0\{\phi^i(x), \partial_\mu\phi^i(x)\}$ a Lagrangian density describing the dynamics of the system. Consider a group of transformations acting on the fields

$$\phi^i(x) \to \phi^i(x) - i\theta^a (T_a)^i_j \phi^j(x), \qquad i = 1, 2, \dots, n, \qquad a = 1, 2, \dots, N, \quad (8.47)$$

where N is the number of generators of G. It equals 1 for the $U(1)$ transformations, 3 for $SU(2)$ and 8 for $SU(3)$, etc. T_a are the matrices of representation to which the fields ϕ^i

belong. For example, if G is $SU(2)$ and ϕ^i's form a doublet, then $T_a = \tau_a/2$, τ_a being the Pauli matrices. If the fields form an isovector, then $(T_a)^i_j = \varepsilon_{aij}$, etc. θ^a are N c-number, infinitesimal, x-independent parameters. For future convenience, we have written the infinitesimal transformations, i.e. we have taken only the first order terms in θ^a. The derivatives of the fields transform like the fields themselves. These transformations are global transformations (also known as the gauge transformations of the first kind). Using Noether's theorem, if we have the invariance of the Lagrangian density under the transformations, we obtain N conserved currents. Again if we want to replace the transformations (8.47) by a group of local transformations,

$$\phi^i(x) \to \phi^i(x) - i\theta^a(x)\,(T_a)^i_j\,\phi^j(x), \tag{8.48}$$

the derivative $\partial_\mu \phi$ will pick up extra term

$$\partial_\mu \phi^i(x) \to \partial_\mu \phi^i(x) - i\theta^a\,(T_a)^i_j\,\phi^j(x) - i\,(T_a)^i_j\,\phi^j(x)\partial_\mu \theta^\alpha(x). \tag{8.49}$$

The last term spoils the invariance of $\mathcal{L}_0$. The prescription for restoring invariance is again the same. We introduce N gauge fields $W^a_\mu(x)$ and define the covariant derivative

$$D_\mu \phi^i(x) \equiv \partial_\mu \phi^i(x) - g\,(T_a)^i_j\,W^a_\mu(x)\,\phi^j(x) \tag{8.50}$$

which transforms like the fields, namely,

$$D_\mu \phi^i(x) \to D_\mu \phi^i(x) - i\theta^a\,(T_a)^i_j\,D_\mu \phi^j(x). \tag{8.51}$$

The gauge fields transform according to

$$W^a_\mu(x) \to W^a_\mu(x) + f^a_{bc} W^c_\mu \theta^b + \frac{1}{g}\,\partial_\mu \theta^a(x) \tag{8.52}$$

with g as a constant and f^a_{bc} as the structure constants of the group, given by

$$[T_b, T_c] = f^a_{bc} T_a. \tag{8.53}$$

If we replace ∂_μ by D_μ in $\mathcal{L}_0$, we get $\mathcal{L}_I$ which is invariant under (8.48) and (8.53). Finally, we add a kinetic energy term for the gauge fields

$$-\frac{1}{4}\,G^a_{\mu\nu} G^{\mu\nu}_a, \tag{8.54}$$

where

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu(x) - \partial_\nu W^a_\mu(x) - g f^a_{bc} W^b_\mu W^c_\nu. \tag{8.55}$$

We note that, unlike the photon, non-Abelian gauge fields are self coupled through $G_{\mu\nu}G^{\mu\nu}$. Further, the gauge fields have still zero mass because a term like $W_\mu W^\mu$ is again non-invariant under (8.52). Since no massless vector bosons other than the photon are known in nature, non-Abelian gauge symmetries have nothing to do with physics in general and weak interactions in particular. To generate masses, spontaneous symmetry breaking can be easily applied here. Suppose we have a Lagrangian density

$$\mathcal{L}(\phi_{i,}, \partial_\mu\phi_i) = T - V(\phi_i), \qquad i = 1, 2, \ldots, n. \tag{8.56}$$

Here ϕ_i are real scalar fields. Assume that $\mathcal{L}$ is invariant under a local group of transformations. Let the elements of the group be represented by $e^{i\alpha_i\tau_i}$, where α^j are real constants and τ are the matrices representing the generators.

$$\phi = \begin{vmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_n \end{vmatrix} \tag{8.57}$$

$\phi \to \phi' = e^{i\alpha_i\tau_i}\,\phi$, $\delta\phi = i\alpha_i\tau_i\phi$ for small α, τ are $n \times n$ matrices, $j = 1, 2, \ldots, n$. Look for the vacuum state by using

$$\frac{\partial V}{\partial \phi_j} = 0. \tag{8.58}$$

Suppose there are several vacua. These will transform into each other under the group. Choose a particular vacuum

$$\phi_0 = \begin{vmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_n \end{vmatrix}_0 \tag{8.59}$$

and disregard the rest. Under an infinitesimal action of the group, the vacuum is changed by the amount

$$\delta\phi_0 = i\alpha_i\tau_i\phi_0. \tag{8.60}$$

We know that the vacuum is then non-invariant under some of the operations of the group, namely those for which

$$T^k\phi_0 \neq 0. \tag{8.61}$$

Suppose there are n' such generators, $k = 1, 2, \ldots, n'$,

$$1 \leq n' \leq n.$$

Then the remaining generators leave the vacuum invariant.

$$T^k\phi_0 = 0, \qquad k = n' + 1, \ldots, n. \tag{8.62}$$

To each generator that breaks the symmetry corresponds a Goldstone boson.

9

THE STANDARD MODEL

9.1 Introduction

The standard model (SM) has been extremely successful in the description of nature so far. There are no conclusive disagreements between the SM predictions and experimental data. As discussed before, the laws of nature are governed by the four fundamental interactions: the strong, the electromagnetic, the weak, and the gravitational. At the atomic and subatomic scale the gravitational interaction is extremely feeble and can be ignored. The other three interactions are described by the SM. The electromagnetic and weak interactions have been unified and the resultant electroweak interaction is described by the Glashow-Weinberg-Salam (GWS) model. The theory of strong interactions is Quantum Chromodynamics (QCD). Together, the GWS and QCD form the standard model.

We know that all the fundamental interactions are carried by specific gauge fields or their equivalent quanta (particles). For example, the electromagnetic interactions are described in terms of photons which are the quanta of the electromagnetic field. It now appears that the strong interactions are transmitted by massless spin 1 gluons, while the weak forces are transmitted by massive spin 1 *W* and *Z* bosons, all closely analogous to the photon of electromagnetism. The quantum treatment of gravitational interaction is still an open question, but the Einstein field equations describing the gravitation indicate that it is transmitted by spin 2 graviton. These forces are all described by the gauge theories, the transmitting particles being gauge bosons. The summary of the fundamental interactions and properties of their gauge bosons is given in Table 9.1.

Table 9.1 Fundamental Forces and Gauge Bosons

Force	*IV B*	*Symbol*	*Charge*	*Spin*	*Mass*
Strong	8 gluons	g	0	1	0
Electromagnetic	Photon	γ	0	1	0
Weak	W, Z	$W^{\pm}, Z^0$	±1, 0	1	81, 92 GeV
Gravitational	Graviton	G	0	2	0

The leptons, quarks and gauge bosons plus the Higgs particles are the basis of the present understanding of the physical world. There are six leptons and six quarks which can be grouped into three *generations* (also called *families*).

9.2 Electroweak Interaction

The first attempt at the unification of the electromagnetic and weak interactions was made by J. Schwinger, in 1956 in which he proposed to put the photon and the charged intermediate bosons W^+ and W^- in a triplet of isospin. S.L. Glashow in 1960 modified the above proposal and put forward an idea that the electromagnetic and weak interactions may be unified in a gauge theory based on the group $SU(2)_L \times U(1)$ thus introducing an additional electrically neutral intermediate vector boson. According to the suggestion of Glashow, the neutral vector boson of the group $SU(2)_L$ and that of the $U(1)_Y$ mix so as to give one linear combination which we identify with photon, whereas the other orthogonal combination is called Z^0 which couples to the neutral weak currents. The model, therefore, necessitates the introduction of the weak neutral current. The low energy phenomenology of weak interaction (Chapter 6) suggests that the weak interaction quanta are massive vector particles. The problem of generating masses consistent with gauge invariance was solved later in 1967 by S. Weinberg and A. Salam, using the idea of spontaneous symmetry breaking through the mechanism due to Higgs. The resulting theory was shown in 1972 by G. t'Hooft to be a renormalizable quantum field theory. The electromagnetic and the weak interactions thus have been unified in the Glashow-Weinberg-Salam model. This model is based on two independent local gauge symmetries: an Abelian gauge symmetry $U(1)_Y$ and a non-Abelian gauge symmetry $SU(2)_L$. Therefore, the model has the total symmetry $SU(2)_L \times U(1)_Y$. The basic quantum numbers of this symmetry group are the electroweak iso-spin T and the electroweak hypercharge Y. To generate the left-handed structure of the charged-current weak interactions, the $SU(2)$ gauge symmetry is applied only to left-handed fermion fields ψ_L. Right-handed fermions are assigned to transform only under $U(1)$.

The electric charge of the leptons and quarks is given by Gell-Mann-Nishijima like relation

$$Q = (T_3)_L + (Y/2)_{L,R} \tag{9.1}$$

T_3 is the third component of the electroweak iso-spin (which is not to be confused with the usual strong isospin because (a) the former is only left handed in contrast to the latter which is left and right handed; (b) this $(T_3)_L$ refers to the Cabibbo mixed particles; and (c) it is defined for leptons also, whereas the strong one is not).

The electroweak quantum numbers of point like Fermi particles (leptons and quarks) of the first generation are shown in Table 9.2. We will have similar tables for the second and third generations. The corresponding properties of antiparticles can be simply obtained by C-conjugation.

Table 9.2 First Generation of Leptons

State	T	T_3	$^1/_2Y$	Q
$(\nu_e)_L$	1/2	1/2	–1/2	0
$(e)_L$	1/2	–1/2	–1/2	–1
$(u)_L$	1/2	1/2	1/6	2/3
$(d)_L$	1/2	–1/2	1/6	–1/3
$(e)_R$	0	0	–1	–1
u_R	0	0	2/3	2/3
d_R	0	0	–1/3	–1/3

The group of transformations generated by T and Y is $SU(2) \times U(1)$. To every generator of each factor of the group, there is associated a massless gauge boson coupling minimally to matter fields. The Higgs' mechanism generates masses for the weak bosons via the spontaneous breakdown of $SU(2) \times U(1)$ to $U(1)_{\text{EM}}$.

9.2.1 Electroweak Theory of Leptons

Every fermion generation contains two related left-handed leptons. The weak charged currents appear to induce transitions within the same families only. Since each lepton family has a charged lepton and its associated neutrino, it is assumed that these form a weak isodoublet denoted by

$$L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{9.2}$$

where

$$e_L = \frac{1}{2}(1 - \gamma_5)e. \tag{9.3}$$

There are also right-handed components of charged leptons. Since neutrinos are only left-handed, the right-handed leptons are represented by singlets, e.g.

$$e_R = \frac{1}{2}(1 + \gamma_5)\, e \equiv R. \tag{9.4}$$

Let us now consider various currents: electromagnetic, charged weak and neutral weak. We start with the electromagnetic current due to electrons only:

$$J_\mu^{\text{EM}} = -\,\bar{e}\gamma_\mu e. \tag{9.5}$$

In terms of the left- and right-handed components, this equation can also be written as

$$J_\mu^{\text{EM}} = -\,\bar{e}\gamma_\mu e = -\,\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R. \tag{9.6}$$

Let us now add and subtract $\frac{1}{2}\,\bar{\nu}_L\gamma_\mu\nu_L$, so that

$$\begin{aligned} J_\mu^{\text{EM}} &= \frac{1}{2}\,(\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L) - \frac{1}{2}\,(\bar{\nu}_L\gamma_\mu\nu_L + \bar{e}_L\gamma_\mu e_L) - \bar{e}_R\gamma_\mu e_R \\ &= \qquad J^3_{\mu(L)} \qquad\qquad + \qquad\qquad J^Y_{\mu(L,R)}, \end{aligned} \tag{9.7}$$

where

$$\left.\begin{aligned} J^3_{\mu(L)} &= \frac{1}{2}\,(\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L), \\ J^Y_{\mu(L,R)} &= -\,\frac{1}{2}\,(\bar{\nu}_L\gamma_\mu\nu_L + \bar{e}_L\gamma_\mu e_L) - \bar{e}_R\gamma_\mu e_R. \end{aligned}\right\} \tag{9.8}$$

$J^3_{\mu(L)}$ consists only of left-handed terms while $J^Y_{\mu(L,R)}$ comprises left- and right-handed terms. $J^3_{\mu(L)}$ is the third component of the electroweak isospin current, whereas $J^Y_{\mu(L,R)}$ is the electroweak hypercharge current. From the coefficients in front of each term, which make up the J_μ, we immediately deduce eigenvalues of T_3 and Y for the states.

$$\left.\begin{aligned} T_3\nu_L &= +\frac{1}{2}\nu_L, & Y\nu_L &= -\nu_L, \\ T_3e_L &= -\frac{1}{2}e_L, & Ye_L &= -e_L, \\ T_3e_R &= 0, & Ye_R &= -2e_R. \end{aligned}\right\} \tag{9.9}$$

We may write

$$J^3_{\mu(L)} = \frac{1}{2}\bar{L}\gamma_\mu\tau_3 L, \qquad J^Y_{\mu(L+R)} = -\frac{1}{2}\bar{L}\gamma_\mu L - \bar{R}\gamma_\mu R.$$

Repeating the same exercise gives us the electroweak quantum numbers for all the leptons and quarks. The gauge fields form an isotriplet W_μ for $SU(2)_L$ and a singlet B_μ for $U(1)$. The term in Lagrangian density for the matter field is

$$\mathcal{L}_{\text{matter}} = \bar{L}i\gamma^\mu D_\mu L + \bar{R}i\gamma^\mu(\partial_\mu + \frac{i}{2}YB_\mu)R. \tag{9.10}$$

The covariant derivative D_μ is given by

$$D_\mu = \partial_\mu + ig\,\frac{1}{2}\,\tau\cdot\mathbf{W}_\mu + ig'YB_\mu. \tag{9.11}$$

The field tensors are

$$\left.\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ F^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon^{abc}W^b_\mu W^c_\nu, \end{aligned}\right\} \tag{9.12}$$

and the transformations of the gauge fields are given by

$$\left.\begin{aligned} B_\mu &\to B_\mu + (1/g')\partial_\mu\Lambda, \\ W^a_\mu &\to W^a_\mu + (1/g)\partial_\mu\Lambda^a + g\varepsilon^{abc}\Lambda^b W^c_\nu. \end{aligned}\right\} \tag{9.13}$$

The interaction Lagrangian density is

$$\mathcal{L}_{\text{int}} = -\frac{g}{2}\bar{L}\gamma_\mu\tau L\cdot\mathbf{W}^\mu + g'[\frac{1}{2}\bar{L}\gamma_\mu L + \bar{R}\gamma_\mu R]B^\mu, \tag{9.14}$$

where L stands for the left-handed lepton doublet and R for the right-handed lepton which is a weak isospin singlet.

The Lagrangian is invariant under the local gauge transformations for $SU(2)_L$ and $U(1)_Y$ independently. W^+, W^- and W^0 are the intermediate boson fields corresponding to the quanta of electroweak isospin group $SU(2)$ coupled to the electroweak isospin and B

is the field coupled to the electroweak hypercharge and Y is a quantum of the electroweak group $U(1)$. The complete symmetry is $SU(2)_L \times U(1)_{L+R}$. While it is clear that the usual charged intermediate bosons can be identified with

$$W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \pm iW^2) \tag{9.15}$$

neither of the two remaining bosons (the neutral ones) is the photon. As suggested by Glashow

$$\left.\begin{aligned} A_\mu &= \cos(\theta_W)B_\mu + \sin(\theta_W)W^3_\mu, \\ Z_\mu &= -\sin(\theta_W)B_\mu + \cos(\theta_W)W^3_\mu. \end{aligned}\right\} \tag{9.16}$$

In order to work out the coupling of the electromagnetic current, we use the following linear combinations of 2 neutral bosons, which are the inverse of Eq. (9.16):

$$\left.\begin{aligned} W^3_\mu &= \cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu, \\ B_\mu &= -\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu, \end{aligned}\right\} \tag{9.17}$$

where θ_W is the Weinberg-Salam mixing angle. Substituting (9.15–9.17) into the Lagrangian density, we get

$$\begin{aligned} \mathcal{L}_{\text{int}} &= -\frac{g}{\sqrt{2}}\bar{L}\gamma_\mu\tau^+ LW^- - \frac{g}{\sqrt{2}}\bar{L}\gamma_\mu\tau^- LW^+ - \frac{g}{2}\bar{L}\gamma_\mu\tau_3 L(\cos(\theta_W)Z^\mu + \sin(\theta_W)A^\mu) \\ &\quad + g'[\frac{1}{2}\bar{L}\gamma_\mu L + \bar{R}\gamma_\mu R](-\sin(\theta_W)Z^\mu + \cos(\theta_W)A^\mu) \\ &= \text{charged part} + \{-g\cos(\theta_W)\frac{1}{2}\bar{L}\gamma_\mu\tau_3 L - g'\sin(\theta_W)(\frac{1}{2}\bar{L}\gamma_\mu L + \bar{R}\gamma_\mu R)\}Z^\mu \\ &\quad + \{-g\sin(\theta_W)\frac{1}{2}\bar{L}\gamma_\mu\tau_3 L + g'\cos(\theta_W)(\frac{1}{2}\bar{L}\gamma_\mu L + \bar{R}\gamma_\mu R)\}A^\mu. \end{aligned} \tag{9.18}$$

We want to identify A^μ with the photon. The A^μ coupling can be written as

$$-\{g\sin(\theta_W)J^3_{\mu L} + g'\cos(\theta_W)J^3_{\mu L})\}A^\mu. \tag{9.19}$$

We then immediately obtain the connection

$$g'\cos(\theta_W) = g\sin(\theta_W) = e \tag{9.20}$$

as the electromagnetic current is $J^3_{\mu L} + J^Y_{\mu(L+R)}$.

9.2.2 Neutral Current

The neutral current, J^{NC}_μ which couples with the neutral intermediate vector boson Z^0, is

$$\begin{aligned}
-\,J_\mu^{\mathrm{NC}} &= \{g\cos(\theta_W)\,J_{\mu L}^3 - g'\sin(\theta_W)\,J_{\mu(L+R)}^Y\}\\
&= \{g\cos(\theta_W)\,J_{\mu L}^3 - g'\sin(\theta_W)\,J_\mu^{\mathrm{EM}} - J_{\mu L}^3\}\\
&= (g\cos(\theta_W) + g'\sin(\theta_W))\,J_{\mu L}^3 - g'\sin(\theta_W)\,J_\mu^{\mathrm{EM}}\\
&= (g_W\cos^2(\theta_W) + g_W\sin^2(\theta_W))\,J_{\mu L}^3 - g_W\sin^2(\theta_W)\,J_\mu^{\mathrm{EM}}\\
&= g_W\,(J_{\mu L}^3 - \sin^2(\theta_W)\,J_\mu^{\mathrm{EM}}).
\end{aligned} \tag{9.21}$$

We have written $g = g_W\cos(\theta_W)$ and $g' = g_W\sin(\theta_W)$. The coupling constants satisfy the following relations

$$g_z = g_W = \frac{e}{\sin(\theta_W)\cos(\theta_W)} = \sqrt{\frac{gg'}{\sin(\theta_W)\cos(\theta_W)}},$$

$$g^2 + g'^2 = g_W^2.$$

The fermion and gauge boson masses are still zero. As explained in Chapter 8, they acquire masses through the Higgs' mechanism in which a local gauge symmetry is broken spontaneously. The gauge symmetry remains valid for the Lagrangian, but not for the ground state. This preserves the renormalizability of the theory.

The weak bosons acquire masses when the $SU(2)_L$ gauge symmetry is broken spontaneously. In the minimal Higgs structure, this is accomplished by introducing a doublet field

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \tag{9.22}$$

with $SU(2)_L$ and $U(1)_Y$ quantum numbers to the Lagrangian. The Higgs' field obtains a nonvanishing vacuum expectation value.

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ v \end{bmatrix}. \tag{9.23}$$

Three of the ϕ degrees of freedom become the longitudinal components of the $W^\pm$ and Z bosons which then acquire the masses by the Higgs' mechanism as

$$M_W = \frac{1}{2}gv, \quad M_z = \frac{1}{2}(g^2 + g'^2)^{1/2}\,v, \tag{9.24}$$

where $v = (\sqrt{2}\,G_F)^{-1/2} = 246$ GeV. One gauge boson remains massless; it is the photon.

The parameter θ_W fixes the masses of charged and neutral weak bosons and predicts that they are different.

$$\left.\begin{aligned} M_W &= 37/\sin(\theta_W)\ \text{GeV},\\ M_z &= 37/\sin(\theta_W)\cos(\theta_W)\ \text{GeV}. \end{aligned}\right\} \tag{9.25}$$

The present experimental values are

$$M_W = 80.33 \pm 0.15 \text{ GeV}, \qquad M_z = 91.187 \pm 0.007 \text{ GeV}.$$

The observation in 1983 of the weak bosons with masses put the electroweak theory on a firm footing, which had already been established in 1974 on the observation of neutral currents at CERN. There still remains one neutral scalar field which manifests itself as a physical particle known as the *Higgs' boson*. The model can be applied to the second and third generations of leptons in a similar manner and as we shall see in Section 9.3, to quarks as well.

9.2.3 *W* and *Z* Widths

The predicted partial width of the W to $e\nu_e$ is

$$\Gamma(W \to e\nu_e) = \frac{G}{\sqrt{2}}\frac{M_W^3}{6\pi} = 226 \pm 12 \,\text{MeV}.$$

The widths to the various channels are expected to be in the ratios

$$e\nu : \mu\nu : \tau\nu : u\bar{d} : c\bar{s} = 1 : 1 : 1 : 3 : 3$$

when we neglect Cabibbo mixing. We expect that

$$\Gamma(Z \to \nu\bar{\nu}) = \frac{G}{\sqrt{2}}\frac{M_Z^3}{12\pi} = 168 \pm 12 \,\text{MeV}.$$

The widths to the various channels are expected to be in the ratios

$$\nu\bar{\nu} : e^+e^- : u\bar{u} : d\bar{d} = 1 : 1 - 4x_w + 8x_w^2 : 3 - 8x_w + \frac{32}{3}x_w^2 : 3 - 4x_w + \frac{8}{3}x_w^2,$$

where $x_w = \sin^2(\theta_w)$.

9.3 Inclusion of Hadrons

We assume that the weak interactions of hadrons occur at the quark level. Since we have lepton quark symmetry, the inclusion of quarks proceeds in a manner similar to leptons. To extend the electroweak theory to hadrons, we assume that quarks like leptons belong to $SU(2) \times U(1)$ representations.

9.4 Weak Quark Mixing

Historically, the first weak interaction process observed was nuclear beta decay. We will see in Section 9.5 as to how we can include quarks and, therefore, hadrons in the Glashow-Weinberg-Salam model. In the quark picture then, the decay $n \to pe^- \bar{\nu}$ can be taken as a transformation of a valence d quark in the neutron into a u quark. The process can be represented by Fig. 9.1.

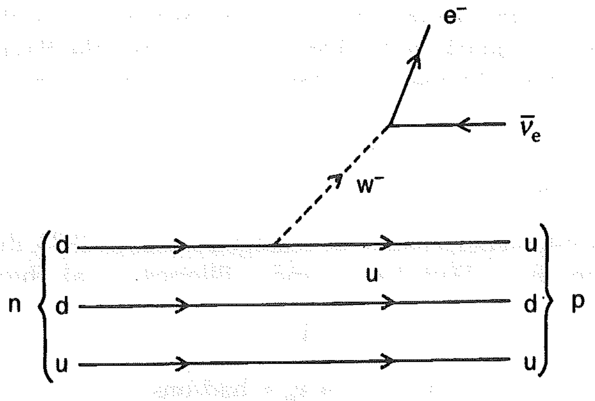

Fig. 9.1 Neutron beta decay in the quark picture.

This process can be explained by a first order perturbation theory based on an effective Lagrangian density of the type

$$\mathcal{L} = \frac{G}{\sqrt{2}} J^{+\alpha} J_\alpha^- + \text{h.c}, \tag{9.26}$$

where J_α^- is the leptonic charged current

$$J_\alpha^- = \bar{e}\gamma_\alpha (1 - \gamma_5)\nu_e \tag{9.27}$$

and $J^{+\alpha}$ is the hadronic charged current

$$J^{+\alpha} = \bar{u}\gamma^\alpha(1 - \gamma_5)dV_{ud}, \tag{9.28}$$

where V_{ud} is the coupling strength of the transition of d to u, compared to that of a pure leptonic process like

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{9.29}$$

Similarly, if we consider the decay of Λ hyperon as the transition of the valence s quark into a u quark, we can express the decay by Fig. 9.2. Experimentally, the coupling strength V_{us} of the corresponding piece of the hadronic current

$$J^{+\alpha} = \bar{u}\gamma^\alpha(1 - \gamma_5)sV_{us}$$

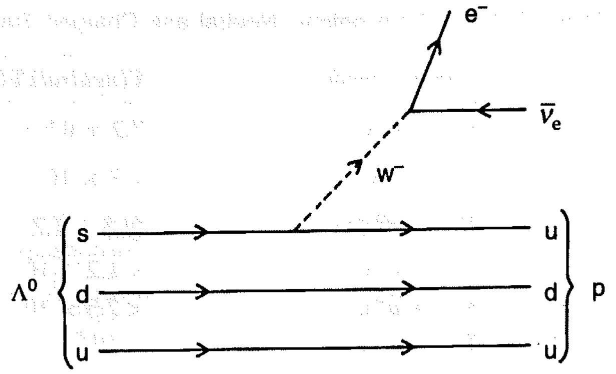

Fig. 9.2 Λ hyperon beta decay in the quark picture.

is about 1/5th of that for the neutron decay V_{ud}. To explain this, Cabibbo postulated that the eigenstates of the two quarks with charge – 1/3 are not the flavour eigenstates d and s but a linear combination obtained by rotation by an angle θ, the Cabibbo angle:

$$d' = d \cos \theta + s \sin \theta$$

9.4.1 Neutral Currents

This picture of weak interactions had to be enlarged when, in 1973, the existence of neutral currents was observed by F. Hasert et al. and J. Blietscau et al. through the reactions

$$\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^-,$$

$$\nu_\mu + N \rightarrow \nu_\mu + \text{hadrons.}$$

Therefore, we have, to add to the leptonic charged current, the current

$$j_\alpha^{NC} = \bar{\nu}\gamma_\alpha(1 - \gamma_5)\nu + \text{terms for } e \text{ and } \mu.$$

This current would require the existence of a massive neutral boson Z^0, together with charged intermediate bosons $W^\pm$. The existence of Z^0, and hence neutral currents, had earlier been postulated with a view to unify weak and electromagnetic interactions via the interference of Z^0 and γ. One problem was left, however. A decay via a neutral current interaction

$$K_L \rightarrow \mu^+\mu^-$$

was searched for, but the experimental branching ratio

$$\frac{\Gamma(K_L \rightarrow \mu^+\mu^-)}{\Gamma(K_L \rightarrow \text{all})} = (7.2 \pm 0.5) \times 10^{-9}$$

is extremely small and can be explained by a second order weak-electromagnetic process (see Table 9.3). Experimentally, it became clear that the strangeness changing neutral currents in the decay $K_L \rightarrow \mu^+\mu^-$ are strongly suppressed. To explain this suppression, S.L. Glashow, J. Illiopoulos and L. Maiani proposed the existence of the c quark before its discovery. This they did to suggest a mechanism which would suppress the $\Delta S = 1, \Delta Q = 0$ weak processes

Table 9.3 Decay Rates of Semileptonic Neutral and Charged Transitions

Neutral modes	*Charged modes*	*Γ(neutral)/Γ(charged)*
$K_L \rightarrow \mu^+\mu^-$	$K^+ \rightarrow \mu^+\nu_\mu$	$7.2 \pm 0.5 \times 10^{-9}$
$K_L \rightarrow e^+e^-$	$K^+ \rightarrow e^+\nu_\mu$	$< 3 \times 10^{-5}$
$K^+ \rightarrow \pi^+e^+e^-$	$K^+ \rightarrow \pi^0e^+\nu_e$	$(4.8 \pm 1.7) \times 10^{-5}$
$K^+ \rightarrow \pi^+\nu\bar{\nu}$	$K^+ \rightarrow \mu^+\nu_\mu$	$< 1.2 \times 10^{-5}$
$K^+ \rightarrow \pi^+\mu^+\mu^-$	$K^+ \rightarrow \mu^+\nu_\mu$	$< 7.5 \times 10^{-5}$
$\Sigma^+ \rightarrow pe^+e^-$	$\Sigma^- \rightarrow ne^-\nu_e$	$< 10^{-2}$

$s \leftrightarrow d$ required to explain the absence of neutral current processes which involve change of strangeness. For example,

$$K^+ \to \pi^+ \nu \bar{\nu}, \qquad K^+ \to \pi^+ e^+ e^-$$

are also absent or highly suppressed relative to the corresponding charged current processes. The diagram (Fig. 9.3) predicts that $K_L \to \mu^+\mu^-$ decay would occur at a rate far in excess of what is observed. The charged current comes from the doublet

$$\begin{pmatrix} u \\ d' \end{pmatrix} = \begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix}. \tag{9.30}$$

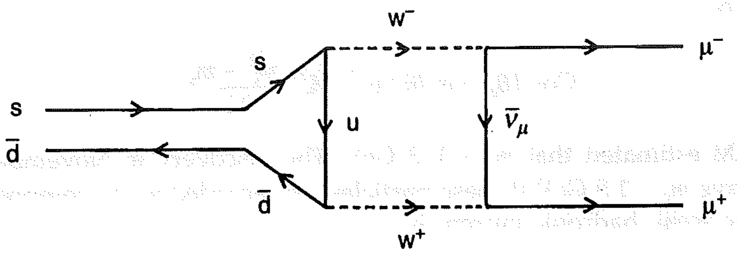

Fig. 9.3 $K_L \to \mu^+\mu^-$.

We can write the neutral current coupling term from this doublet as

$$u\bar{u} + (d\bar{d}\cos^2\theta_c) + (s\bar{s}\sin^2\theta_c) + (s\bar{d} + d\bar{s})\sin\theta_c\cos\theta_c. \tag{9.31}$$

This hadronic neutral current contains, besides strangeness conserving ($\Delta S = 0$) terms which convert a u quark into a u quark, a d into d or s into s, also a strangeness changing ($|\Delta S| = 1$) term converting d into s or vice-versa, which we do not want. The Glashow-Iliopoulos-Maiani proposal cancels the $|\Delta S| = 1$ neutral current term with a new term arising from the existence of a new quark, c. Since $d' = d\cos\theta_c + s\sin\theta_c$ is one linear combination of two quarks d and s, which have the same charge and so can mix in the weak interaction because strangeness is not conserved, there is also an orthogonal combination $s' = s\cos\theta_c - d\sin\theta_c$. We should then have an additional doublet

$$\begin{pmatrix} c \\ s' \end{pmatrix} = \begin{pmatrix} c \\ s\cos\theta_c - d\sin\theta_c \end{pmatrix}.$$

This doublet adds a new term

$$c\bar{c} + (d\bar{d}\sin^2\theta_c + s\bar{s}\cos^2\theta_c) - (s\bar{d} + d\bar{s})\sin\theta_c\cos\theta_c$$

to the neutral current which becomes

$$(u\bar{u} + c\bar{c}) + (d\bar{d} + s\bar{s})(\cos^2\theta_c + \sin^2\theta_c) + (s\bar{d} + d\bar{s})(\sin\theta_c\cos\theta_c - \sin\theta_c\cos\theta_c) \tag{9.32}$$

so that the strangeness changing term cancels to zero. In terms of the GIM mechanism (Fig. 9.4), the new quark c contributes. This diagram has the same magnitude but opposite

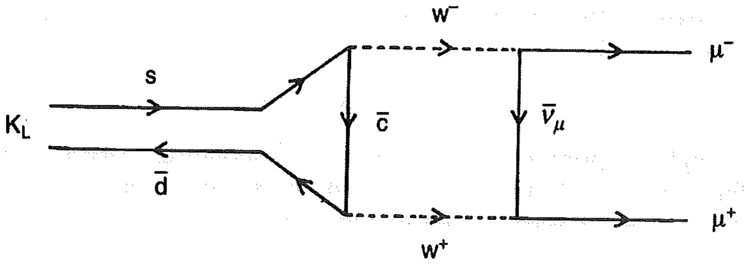

Fig. 9.4 GIM mechanism.

sign to the earlier one and thus cancels it to zero. The calculation shows that the amplitude is of the order of

$$\mathrm{Cos}\ (\theta_c)\ \sin(\theta_c)\ (g^2/M_w^2)\ \frac{m_c^{\prime 2} - m_u^2}{M_w^2}$$

from which GIM estimated that $m_c \sim$ 1–3 GeV. The discovery in November 1974 of the J/Ψ particles gave $m_c \sim 1.5$ GeV if these particles are regarded as $c\bar{c}$ composites. Then for four quarks, the weak hadronic current is

$$J^{+\alpha} = (\bar{u}\ \bar{c}) \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \gamma^\alpha (1 - \gamma_5) \begin{bmatrix} d \\ s \end{bmatrix}, \tag{9.33}$$

where the 2 × 2 unitary matrix is simply a rotation matrix with the rotation angle θ_c, the Cabibbo angle. There is no room here for the *CP* violation. The absence of the strangeness changing neutral currents thus leads to a picture in which the mass eigenstates d and s are mixed to give us the weak eigenstates d' and s' through the relation

$$\begin{bmatrix} d' \\ s' \end{bmatrix} = \begin{bmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} = U \begin{bmatrix} d \\ s \end{bmatrix}. \tag{9.34}$$

We may remark here that we have the freedom of mixing of either upper or lower components of the weak doublets. We have chosen to mix lower components. We can write the above equation as

$$d'_{ij} = U_{ij}\, d_j. \tag{9.35}$$

with $d_1 = d, d_2 = s;\ d'_1 = d', d'_2 = s'$. Note that the unitarity of U forbids the flavour changing $s \to d$ transition. Let us note that the most general U can be written as

$$U = \begin{bmatrix} \cos\theta_c\, e^{i\alpha} & \sin\theta_c\, e^{i\gamma} \\ -\sin\theta_c\, e^{-i\gamma} & \cos\theta_c\, e^{-i\alpha} \end{bmatrix} \tag{9.36}$$

so that

$$\left.\begin{aligned} d' &= d\cos\theta_c\ e^{i\alpha} + s\sin\theta_c\ e^{i\gamma} = e^{i\alpha}\,[d\cos\theta_c + s\sin\theta_c\ e^{i(\gamma-\alpha)}], \\ s' &= -\,d\sin\theta_c\ e^{-i\gamma} + s\cos\theta_c\ e^{-i\alpha} = e^{-i\gamma}\,[-\,d\sin\theta_c + s\cos\theta_c\ e^{i(\gamma-\alpha)}]. \end{aligned}\right\} \tag{9.37}$$

We can redefine the quark states absorbing phases without changing any observable. We redefine $d' e^{-i\alpha}$, $s'e^{i\gamma}$ and $se^{-i(\gamma-\alpha)}$. Let us consider the generalization of the above considerations to N-doublets of left-handed quarks.

$$d'_{ij} = U_{ij}d_j, \qquad i, j = 1, 2, \ldots, N. \tag{9.38}$$

d'_i are mixtures of mass eigenstates d_i. U is a $N \times N$ unitary matrix. Such a matrix has, in general, N^2 parameters. We want to find the number of observable independent parameters in U. We can redefine each of the $2N$ quark states using the phase transformations without changing the physics. We have then $2N$ unphysical phases. One phase has to be omitted, however, as an overall phase. In other words, $2N - 1$ parameters can be absorbed in redefining quark wave function. Hence the actual number of parameters is not N^2 but $N^2 - (2N - 1) = (N - 1)^2$. Of these, $N(N - 1)/2$ can be chosen as rotation angles because an orthogonal $N \times N$ matrix has $N(N - 1)/2$ real parameters. We have, for example, one parameter (the Cabibbo angle) for 2×2 matrix and 3 for 3×3 case. The considerations of the relative phases of the quark states reveal, therefore, that in general, we cannot redefine all the phases to make the U matrix real. For the $N \times N$ matrix there will remain $(N - 1)^2 - N(N - 1)/2 = (N - 1)(N - 2)/2$ residual phases. So to have a residual phase, we need $N > 2$.

With the observation of fifth quark b in 1977, the current had to be extended. M. Kobayashi and T. Maskawa had earlier realized that when U becomes 3×3 matrix, there are three rotations and one complex phase as parameters. That provides an explanation for the CP violation.

For the three generations of quarks, the quark states $|d'\rangle$, $|s'\rangle$, and $|b'\rangle$ result from the mixing of the mass eigenstates $|d\rangle$, $|s\rangle$, and $|b\rangle$ known from the strong interactions by a unitary transformation (*generalization of the GIM mechanism*) as do the corresponding operators.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \tag{9.39}$$

where

$$U = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{9.40}$$

One then also talks of the universality of weak interactions, since this allows the coupling constants of all hadron charges to remain the same.

A unitary 3×3 matrix has nine free real parameters. Five of these can be used in the redefinition of six quarks wavefunctions by the absorption of five phase factors. We are then left with four real parameters. Kobayashi and Maskawa use as parameters three Euler angles θ_1, θ_2 and θ_3 for a real three-dimensional rotations and there is, in addition, one phase for the coupling of s and b quarks. Then we have

The KM matrix is unitary and can be parametrized in various ways. The "standard" parametrization[1] of the matrix is

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}\,e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{9.41}$$

where

$$C_{ij} = \cos\,(\theta_{ij}), \qquad s_{ij} = \sin\,(\theta_{ij}) \tag{9.42}$$

for the generation labels $i, j = 1, 2, 3$. The rotation angles are defined in a way which relates to the mixing of two specific generations of quarks.

There is an arbitrary phase which makes the Lagrangian density nonreal. Thus the Lagrangian density does not obey time-reversal invariance. Then by CPT theorem, it violates CP invariance. We have, therefore, the possibility of accommodating CP violation in the three-generation model.

9.5 Quantum Chromodynamics

In the 1950s and 1960s it was generally believed that strong interactions of hadrons could not be described in any sense by perturbation methods of field theory. A theory known as Quantum Chromodynamics (QCD) has now become a candidate theory to describe hadrons. The theory of QCD was developed by H. Fritzch and M. Gell-Mann in 1972. The principal breakthrough came in 1973 when the property of asymptotic freedom of non-Abelian gauge theory was discovered by G. t'Hooft, D. J. Gross and H.D. Politzer. The property of asymptotic freedom now allows perturbative treatment of strong interactions at high energies and at short distances. The theory has had a number of qualitative and semi-quantitative successes. These successes have given rise to some hope that, finally, we may have a theory of strong interactions, but further theoretical work as well as further experimental tests are needed.

Quantum chromodynamics is a theory in which hadrons are assumed to be composite particles, their constituents being quarks. But QCD goes beyond the quark model of Gell-Mann and Zweig in being a dynamical theory of strong interactions of quarks. When in the 60s the classification of hadrons and hadron masses suggested that hadrons were made up of quarks, it was natural to look for the dynamics obeyed by the quark systems. It appeared from the deep inelastic scattering of electron by protons that protons behave as if they were composed of constituents (called partons by Feynman) which moved independently of one another inside the proton. In other words, when proton is probed at high energy and high momentum transfer, it appears to be made of constituents which act very nearly like free particles (see Chapter 6). The dynamics governing the parton systems should have then the property that interaction between partons, later identified with quarks, becomes weaker at short distances. When G. t'Hooft, D.J. Gross, F. Wilczek and H.D. Politzer found that the non-Abelian gauge field theory have the property called asymptotic freedom, it became natural to look for the non-Abelian gauge field theory for the dynamics governing the quark systems. In the meantime, it had been suggested that quarks must have new quantum number called colour in order to resolve certain difficulties in the quark model. Fritzsch and Gell-Mann in 1972–73 proposed that the extra symmetry required by the non-Abelian gauge field theory be identified with this new quantum number colour. So the theory was named quantum chromodynamics in analogy with the Abelian gauge field theory of electomagnetic interactions, the Quantum Electrodynamics (QED). The gauge field mediating the colour interaction is called the *gluon,* as it is responsible for binding the quarks together to form hadrons. While photons, the gauge field mediating the electromagnetic interactions between charged particles have no electric charge, gluons carry colour charges and hence interact with each other unlike photons. It is this property of self-interacting gluons which makes chromodynamics different from electrodynamics and is essentially responsible for the asymptotic freedom. Like QED, we want a theory of coloured quarks to be invariant under local gauge transformations. Since there are three colours, the gauge group is $SU(3)$ and the gauge transformations involve three colour degrees of freedom of quarks. The generators of the group are the eight Gell-Mann matrices. There are thus eight massless gauge bosons called gluons. Three coloured quarks corresponding to each quark flavour form a triplet in the fundamental representation of $SU(3)$ and eight gluons form an octet in the adjoint representation.

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[1] C. Caso et al. (1998): Particle Data Group, Review of Particle Properties. *The European Physical Journal*, **C3**.

The QCD Lagrangian density is

$$\mathcal{L} = \bar{\psi}_i (i\gamma_\mu D^\mu_{ij} - \delta_{ij} M)\psi_j - \frac{1}{4} F^{\mu\nu}_c F_{c\mu\nu}, \tag{9.43}$$

where i, j, c are colour indices and take the values $i, j = 1, 2, 3$ and $c = 1, \ldots, 8$; M is the quark mass. The covariant derivative D acting on the quark field is

$$D^\mu_{ij} = \delta_{ij}\partial^\mu + i g_s (T_c)_{ij} G^\mu_c, \tag{9.44}$$

where G^μ_c are the gluon fields, T_c the generators of $SU(3)$, and g_s is the strong coupling constant. The gluon field tensor is

$$F^{\mu\nu}_c = \partial^\mu G^\nu_c - \partial^\nu G^\mu_c - g_s f_{\mathrm{cab}} G^\mu_a G^\nu_b, \tag{9.45}$$

where f_{cab} are the structure constants of $SU(3)_c$. The Lagrangian density is invariant under the local gauge transformation

$$\psi(x) \to \exp\,(-\,i g_s \alpha_c(x) T_c)\,\psi(x), \tag{9.46}$$

$$G^\mu_c(x) \to G^\mu_c(x) + \partial^\mu \alpha_c(x) + g_s f_{\mathrm{cab}} \alpha_a(x) G^\mu_b(x). \tag{9.47}$$

Because of the non-Abelian nature of the QCD, it has very different features from the QED. Besides the quark-gluon interaction vertex, which has an analog in QED, it contains three gluon and four gluon vertex interactions that have no analog in QED. In the case of non-Abelian gauge theories, such as the QCD, certain vertex corrections can be summed to all orders, with the result that the coupling constant becomes a function of the mass of the gluon attached to the vertex and is referred to as the running coupling constant. Specifically, the running coupling constant, which plays the role of an effective coupling constant, is given in perturbation theory by

$$\alpha_s(q^2/\mu^2) = \frac{g^2/4\pi}{1 + b(g^2/4\pi)\ln(q^2/\mu^2)} + O(\alpha_s^2), \tag{9.48}$$

where $g^2/4\pi$ is the physical coupling constant, μ^2 is a mass scale introduced due to coupling constant renormalization, and the parameter b depends on the group structure. For $SU(3)_c$,

$$b = (1/12\pi)\,(33 - 2N_f), \tag{9.49}$$

where N_f is the number of flavours; b is positive so long as we do not have more than 16 flavours. The effective coupling goes to zero logarithmetically and the theory is asymptotically free. The opposite happens in QED as well as in any Abelian theory, where b is negative.

In the small r domain, which corresponds to large momenta, the effective coupling constant is small so that we can apply perturbation theory. There is then analogy between one-gluon and one-photon exchange. So we have a potential which looks like the Coulomb potential at small distances but which grows at large distances.

9.5.1 Non-perturbative Regime of QCD and Confinement

As the energy of interaction decreases (the distance scale of the interaction increases), the strength of interaction between quarks and gluons or gluons and gluons becomes large and perturbation calculations can no longer be done. This leads to the phenomenon called confinement; quarks cannot be observed outside of hadrons.

Although very high energy probes demonstrate abundant evidence that hadrons are composed of quarks, we do not see quarks in free state in physical processes. The problem of confinement in QCD is a difficult one. Though not proved so far, confinement is a plausible hypothesis and assumed to be a property of QCD.

In fact, we are faced with a dilemma that in the regime where QCD is calculable (perturbative QCD at short distances), we have only a few experimental tests available, whereas in the regime (non-perturbative QCD, low energy and long distance) where we have a lot of data, perturbative calculations cannot be carried out and we are unable to satisfactorily calculate the confinement effects. There are, however, suggestive and useful models which permit us to understand properties of hadrons at low energies.

PROBLEMS

1. Compute the number of real parameters of an $n \times n$ orthogonal matrix and the number of independent parameters of an $n \times n$ unitary matrix. Then obtain the residual phases. Give the physical significance of the phase in the case of $n = 3$.
2. Show that the width for $w^- \to e\bar{v}_e$ is given by

$$\Gamma(w^- \to e\bar{v}_e) = \frac{GF}{\sqrt{2}}\,\frac{M_w^3}{6\pi}$$

and that for $Z^0 \to v\bar{v}$ by

$$\Gamma(Z^0 \to v\bar{v}) = \frac{GF}{\sqrt{2}}\,\frac{M_z^3}{12\pi}.$$

10

CHARM AND HEAVIER FLAVOURS

10.1 Introduction

Until 1974, all the known mesons and baryons could be understood as quark, antiquark and three quark bound states respectively, where quarks come in three varieties referred to as "flavours" and commonly called *u*, *d*, and *s*. As discussed in Chapter 5, *u* and *d* (up and down quarks) form an isospin doublet and are the constituents of nucleon and pion. The heavier strange *s* (strange) quark can bind with the lighter *u* and *d* quarks as with itself to form the so-called strange particles that make the $SU(3)$ multiplets. The relative heaviness of the strange quark implies that the $SU(3)$ symmetry is broken. With the success of $SU(3)$ in the 1960s, many people thought the possibility that yet heavier quarks might exist. One of the reasons for such thinking was the idea of the lepton-quark symmetry. There were then four leptons—the electron, the muon and their respective neutrinos. Then why not the fourth quark? J.D. Bjorken and S.L. Glashow, in fact, suggested in 1964, the existence of four rather than three fundamental quark fields, and noted that this additional charm quark would lead to a more symmetrical situation between quarks and leptons. In addition, there was a problem in the weak interaction phenomenology—the absence of strangeness changing neutral currents—that led Glashow, Iliopoulos and Maiani (GIM) to develop the idea considerably further. They proposed a weak interaction model (the GIM model) which, by incorporating the charm quark, explained the experimental smallness of strangeness changing neutral currents, as discussed in Chapter 9. M.K. Gaillard and B.W. Lee estimated its mass to be nearly 1.5 GeV/c^2.

In November 1974, at both Brookhaven and SLAC, hidden charm was discovered in the form of a narrow resonance of mass 3.1 GeV/c^2 with an extremely narrow width $\Gamma \cong 70$ keV. This resonance was named J by the Brookhaven group and ψ by the SLAC group; it is usually referred to as J/ψ. The J/ψ particle is interpreted as one of the energy levels of a bound system of the charmed quark and its antiparticle similar to positronium and to the $s\bar{s}$ system. The $c\bar{c}$ system is called a charmonium in analogy with positronium which is a bound system of electron and positron. Several states of charmonium have been observed. The charmonium states have zero value for the new flavour charm carried by c-quark and $\bar{c}$ (antiquark). That is why the $c\bar{c}$ states are sometimes said to have "hidden" charm.

We will not discuss in detail the properties of charmonium (or, in general, quarkonium) states. For further details, the reader may refer to other books or review articles.

10.2 Discovery of Charm

The J/Ψ was seen in 1974 in the reaction

$$p + Be \to e^- e^+ + X$$

at $\sqrt{s} = 7.6$ GeV and also in the processes

$$\left.\begin{aligned} e^-e^+ &\to e^-e^+, \\ e^-e^+ &\to \mu^-\mu^+, \\ e^-e^+ &\to \text{hadrons} \end{aligned}\right\} \tag{10.1}$$

at $\sqrt{s} = 3.105$ GeV. It was observed as a very narrow resonance at the invariant e^-e^+ mass $m(e^-e^+) = 3.1$ GeV. The resonant cross-section for any final state f is given by the Breit-Wigner formula

$$\sigma = \frac{\pi}{k^2} \frac{2J+1}{(2s_1+1)(2s_2+1)} \frac{\Gamma_e \Gamma_f}{(\sqrt{s}-m)^2 + (\Gamma^2/4)} \tag{10.2}$$

where, J is the spin of the resonant particle, m its mass, s_1 and s_2 are spins of the incident particles, and $s = E_{cm}^2 = 4(k^2 + m^2) \approx 4k^2$. Therefore, the process

$$e^-e^+ \to J/\Psi \to f,$$

the cross-section is given by

$$\sigma = \frac{\pi}{s}(2J+1) \frac{\Gamma_e \Gamma_f}{(\sqrt{s}-m)^2 + \Gamma^2/4}. \tag{10.3}$$

For a narrow resonance, $\dfrac{1}{(\sqrt{s}-m)^2 + (\Gamma^2/4)}$ in (10.3) can be replaced by

$$\frac{2\pi}{\Gamma}\,\delta(\sqrt{s}-m).$$

yielding

$$\int \sigma_f d\sqrt{s} = \pi(2J+1)\frac{2\pi\Gamma_e\Gamma_f}{m^2\Gamma}. \tag{10.4}$$

Since $\sigma_{\text{tot.}} = \sum\limits_f \sigma_f$ and $\Gamma = \sum\limits_f \Gamma_f$, taking $\Gamma_e = \Gamma_\mu$, we obtain, for the process, the relation

$$e^-e^+ \to J/\Psi \to \mu^-\mu^+$$

from (10.4)

$$\int \sigma_f d\sqrt{s} = 2\pi^2(2J+1)\frac{\Gamma_e^2}{m^2\Gamma} \tag{10.5}$$

and

$$\sum_f \int \sigma_f d\sqrt{s} = 2\pi^2(2J+1)\frac{\Gamma_e}{m^2} \tag{10.6}$$

If $J = 1$, (10.5) and (10.6) can give us the width Γ_e and total width Γ. Also, because $\Gamma = \Gamma_e + \Gamma_\mu + \Gamma_h$, we can determine hadronic decay width Γ_h. The experimental values turn out to be

$$\Gamma_e = \Gamma_\mu = 4.72 \text{ keV} \pm 0.35, \qquad \Gamma = 68 \pm 10 \text{ keV}.$$

10.2.1 Spin-parity of J/Ψ

The study of the interference between $e^-e^+ \to \gamma \to \mu^-\mu^+$ and $e^-e^+ \to J/\Psi \to \mu^-\mu^+$ allows us to determine the spin parity of the resonance J/Ψ. The cross-section for the process

$$e^-e^+ \to \gamma \to \mu^-\mu^+$$

is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta), \qquad \sigma = \frac{4\pi\alpha^2}{s}.$$

If the spin parity of J/Ψ is 1^-, the same as that of photon, then the angular distribution would not change by the interference between the photon and resonance amplitudes. In fact, experimentally, it was found to be $(1 + \cos^2\theta)$ near the resonance, so that the spin parity of the resonance is indeed 1^-.

10.3 Charmed States

The observations on the charmonium states are strong indications for the existence of the charm quark, but the real proof required direct observation of mesons and baryons made from a charm quark and light quarks. The direct evidence for the existence of charmed particle was first announced by G. Goldhaber et al. and I. Perruzzu et al. in 1976, a year after the discovery of J/ψ. The $l = 0$ mesons which have been observed are $D^0 = c\bar{u}$, $D^+ = c\bar{d}$, $\overline{D}^0 = \bar{c}u$, $D^- = \bar{c}d$, $D_s^+ = c\bar{s}$ and $D_s^- = \bar{c}s$. In fact, the advent of the charmed quark gave rise to a complete new spectroscopy of mesons (Table 10.1) and baryons having c and $\bar{c}$ as their constituents.

10.4 Charm Particle Decays

The charm particle decays are characterized by a subtle interplay of the weak and strong interactions. The weak decays of particles containing a heavy quark can be described in a simple spectator model which treats the heavy quark as essentially a free particle. In the case of ordinary particles containing only light quarks, the decay involves strong interaction bound state effects which we do not know how to calculate. The charm quark could only be taken as medium heavy and thus a complete picture of its decay must incorporate the weak and strong interactions.

10.4.1 The Naive Spectator Model

The momentum transfer in weak decays of heavy, i.e. charm hadrons, is of the order of heavy quark mass. It is, therefore, not unreasonable to assume that the decay of heavy

Table 10.1 15 ⊕ 1 Mesons

SU(4) *repres.*	*SU*(3) *repres.*	*SU*(2) *repres.* I	*Number of states*	*Wavefunction in quark flavours*
		1	$3(\pi)$	$\bar{d}u, \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d), \bar{u}d$
		1/2	$2(K)$	$\bar{s}u, \bar{s}d$
	8	1/2	$2(\bar{K})$	$\bar{d}s, \bar{u}s$
		0	$1(\eta_8)$	$\frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$
15		1/2	$2(D)$	$\bar{d}c, \bar{u}c$
	3	0	$1(D_s)$	$\bar{s}c$
		1/2	$2(\bar{D})$	$\bar{c}u, \bar{c}d$
	$\bar{3}$	0	$1(\bar{D}_s)$	$\bar{c}s$
	1	0	η_1	$\frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s)$
1	1	0	η_c	$\bar{c}c$

flavour involves the dynamics at short distances. As a consequence of asymptotic freedom in QCD, the heavy quark can then be treated as essentially a free particle. Thus the decay of charm mesons can be described by the decay of the charm quark with the light constituents acting as passive spectators. In this spectator model picture, the decay is expressed as in Fig. 10.1.

The essential assumptions of the spectator model are:

(i) The charm meson is represented by its valence quark configuration, i.e. a charm quark and a light antiquark only. All complex bound state fluctuations are disregarded.

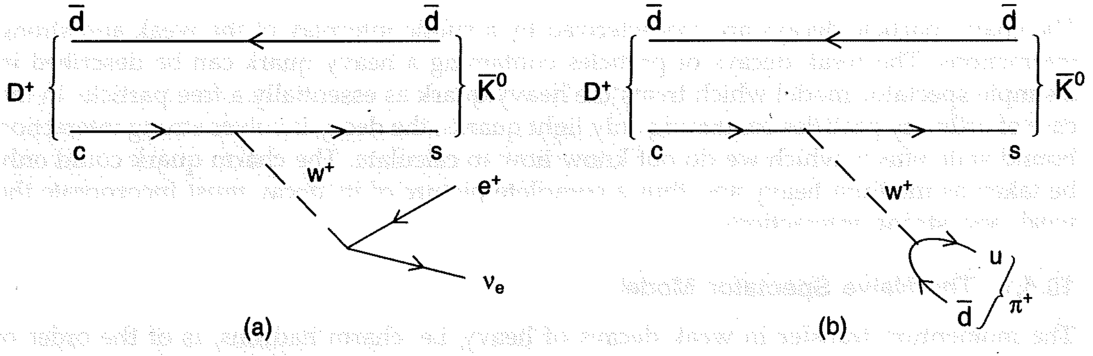

Fig. 10.1 Spectator model for charm meson decay.

(ii) Soft gluon emissions are neglected.

(iii) The sum of all possible hadronic final states is replaced with final states of free quarks emitted in the decay.

The charm meson lifetime is then readily calculated. The matrix element of D meson into an arbitrary nonleptonic (NL) hadronic final state X_H is given by

$$M = \langle X_H | (\bar{u}d)\,(\bar{s}c) | D \rangle, \tag{10.7}$$

where the notation $(\bar{q}_1 q_2) = \bar{q}_{1i}\gamma^\mu (1 + \gamma_5)\, q_{2i}$ (i being the colour index) has been used. With the help of assumption (iii), we can write

$$M = \langle \bar{q}su\bar{d} | (\bar{u}d)(\bar{s}c) | D \rangle . \tag{10.8}$$

Application of assumption (i) gives

$$M \approx \langle su\bar{d} | (\bar{u}d)(\bar{s}c) | c \rangle = (\bar{u}_u \gamma^\mu (1 - \gamma_5) v_{\bar{d}})(\bar{u}_s \gamma_\mu (1 - \gamma_5) u_c). \tag{10.9}$$

The matrix element of c quark decay, represented by Fig. 10.2, is formally identical to the μ decay matrix element so that we can write for the partial width associated with such a process

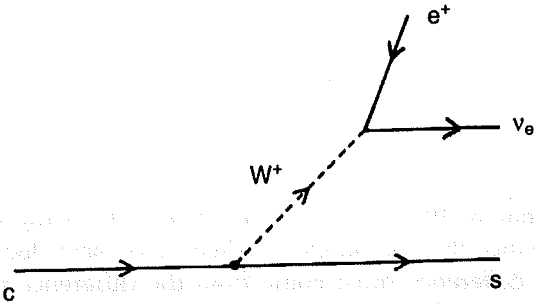

Fig. 10.2 Beta decay of a free charm quark.

$$\Gamma_0 (c \to su\bar{d}) = \frac{G_F^2 M_c^5}{192\pi^3} \tag{10.10}$$

There are two Cabibbo-allowed semi-leptonic, and three Cabibbo allowed non-leptonic modes corresponding to three colours

$$\Gamma^{\text{tot}} = \Gamma^{\text{NL}} + \Gamma^{\text{SL}}, \tag{10.11}$$

where

$$\Gamma^{\text{NL}} = 3 \times \Gamma_0, \tag{10.12}$$

$$\Gamma^{\text{SL}} = 2 \times \Gamma_0. \tag{10.13}$$

The D-meson lifetime is then

$$\tau_D = \frac{1}{\Gamma^{\text{NL}} + \Gamma^{\text{SL}}} = \frac{1}{5}\left(\frac{m_\mu}{m_c}\right)^5 \tau_\mu \approx 9 \times 10^{-13}\,\text{s}. \tag{10.14}$$

As is clear from the above calculation, an important prediction of the spectator model is that since the charm quark decay is independent of the light quarks in the charm hadron, then all the charm particles will have the same lifetime, i.e.

$$\tau_{D^+} = \tau_{D^0} = \tau_{D_s} = \tau_{\Lambda_c^+} = \ldots \tau_c. \tag{10.15}$$

Another prediction of the spectator model is that all charm particles have the same semi-leptonic branching ratio

$$\mathrm{BR}_l\,(D^+) \approx \mathrm{BR}_l\,(D^0) \approx \mathrm{BR}_l\,(D_s^+) \approx \mathrm{BR}_l\,(\Lambda_c^+) = \mathrm{BR}_l\,(c). \tag{10.16}$$

The current measurements of the branching ratios give

$$\mathrm{BR}(D^0 \to e^+ \nu_e X) = 7.7 \pm 1.2\%, \tag{10.17}$$

$$\mathrm{BR}(D^+ \to e^+ \nu_e X) = 17.2 \pm 1.9\%, \tag{10.18}$$

where X represents one or more hadrons. Since

$$\Gamma^{\mathrm{SL}} = \left(\frac{\Gamma^{\mathrm{SL}}}{\Gamma_D}\right) \times \Gamma_D = \mathrm{BR}\,(D \to e^+ X)/\tau_D, \tag{10.19}$$

we have

$$\Gamma^{\mathrm{SL}}_{D^0} = (1.82 \pm 0.30) \times 10^{13}\ \mathrm{s}^{-1}, \tag{10.20}$$

$$\Gamma^{\mathrm{SL}}_{D^+} = (1.63 \pm 0.20) \times 10^{13}\ \mathrm{s}^{-1}. \tag{10.21}$$

The semi-leptonic partial widths are thus consistent with being equal, as the spectator model would predict, since there is no QCD influence on semi-leptonic decay transitions. Therefore, the lifetime difference must come from the difference in nonleptonic widths.

In the spectator model, these two branching ratios should be equal. There has been considerable effort to understand charm hadrons by including nonspectator diagrams.

10.5 Corrections to the Spectator Picture

Although the spectator model is aesthetically appealing due to its simplicity, it is perhaps unreasonable to expect that the model would provide complete description of charm particle decays. This is because a real meson is a superposition of quark and gluon configurations, where the valence quark configuration used in the spectator model is the simplest. The nonleptonic weak decays are expected to be modified by the strong interactions among the quarks. The strong interaction corrections are of two kinds: (i) soft gluon radiative corrections, and (ii) hard gluon radiative corrections.

10.5.1 Soft Gluon Corrections

The calculation of lowest order soft gluon radiative corrections is identical to the em radiative corrections to μ decay with the substitution of em coupling constant α by the strong interaction coupling constant α_s,

$$\frac{\alpha}{\pi} \rightarrow \frac{1}{3} Tr\left(\frac{\lambda^\alpha}{2}\frac{\lambda^\alpha}{2}\right)\frac{\alpha_s}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}, \tag{10.22}$$

which has been averaged and summed over initial and final colour states.

For simplicity, let us consider only the corrected semi-leptonic width

$$\Gamma_{\mathrm{SL}}^{\mathrm{corr}} = (\Gamma_{\mathrm{SL}})\, g(\varepsilon)\left[1 - \frac{2}{3}\frac{\alpha_s(m_c^2)}{\pi} f(\varepsilon)\right], \tag{10.23}$$

whereas earlier

$$\Gamma_{\mathrm{SL}} = \Gamma_0 = \frac{G_F^2 m_c^5}{192\pi^3} \tag{10.24}$$

was the uncorrected width and $g(\varepsilon)$ a mass correction for finite strange quark mass

$$g(\varepsilon) = 1 - 8\varepsilon^2 - 24\varepsilon^4 \ln \varepsilon + 8\varepsilon^6 - \varepsilon^8, \tag{10.25}$$

where $\varepsilon = m_s/m_{cr}$ and $f(\varepsilon)$ is a smooth decreasing function of ε bounded by

$$f(0) = \pi^2 - 25/4, \qquad f(1) = 3/2, \tag{10.26}$$

$f(0)$ corresponds to μ decay result. The value of $\Gamma_{\mathrm{SL}}^{\mathrm{corr}}/\Gamma_{\mathrm{SL}}$ for $\varepsilon = 0$ is of the order of 0.50. The semi-leptonic branching ratio is, however, not largely affected because a large part of the correction to $\Gamma_{\mathrm{SL}}^{\mathrm{corr}}$ exactly cancels the effect in the hadronic channel. The overall effect on the spectator model prediction is to raise the lifetime by a factor of 3 and lower $\mathrm{BR}_l(c)$ by ~ 30%.

10.6 Nonleptonic Transitions

The nonleptonic Hamiltonian in the spectator model for the process $c \rightarrow su\bar{d}$ can be written as

$$H_{\mathrm{NL}} = \frac{G_F}{\sqrt{2}} \cos^2\theta_c\, (\bar{s}c)\, (\bar{u}d). \tag{10.27}$$

$(\bar{q}_i q_j) \equiv \bar{q}_i \gamma_\mu (1 - \gamma_5) q_j$. This four-fermion point interaction is altered in lowest order by one loop hard gluon exchange. The effective weak Hamiltonian for the nonleptonic interaction can be written as

$$H_{\mathrm{weak}}^{\mathrm{NL}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud}\, (c_+ O_+ + c_- O_-), \tag{10.28}$$

where

$$O_\pm = \frac{1}{2}\left[(\bar{u}d)\,(\bar{s}c) \pm (\bar{s}d)\,(\bar{u}c)\right]. \tag{10.29}$$

The Wilson coefficients $c_\pm(q^2)$ contain strong interaction effects. $O_-(O_+)$ transform like an $SU(4)$ 20″-plet (84-plet). In the charm-changing piece, these contain respectively the $SU(3)$ 6*-plet and 15-plet. $c_\pm$ are given by

$$c_- = c_+^{-2} = \left[\frac{\alpha_s(m_c^2)}{\alpha_s(m_W^2)}\right]^{[12/(33-2N_f)]} \tag{10.30}$$

When $c_+ = c_- = 1$, there is no enhancement, and the $H_{\rm NL}$ reduces to what we had before. $c_\pm$ have been estimated and their values are $c_+ = 0.74$, $c_- = 1.8$. These numbers lead to a nonleptonic enhancement of $2 + 2c_+^2 + c_-^2 = 4.34$ compared to earlier factor of 3. The semileptonic branching ratio becomes

$$\mathrm{BR}_l(c) \to \frac{1}{2 + 2c_+^2 + c_-^2} \approx 16\% \tag{10.31}$$

compared to the naive spectator model prediction of 20%.

10.7 Other Processes

The naive spectator model, even with the strong interaction corrections, does not produce differences in D meson lifetimes. In order to produce these differences, other processes like the nonspectator diagrams and final state interactions should be included.

10.8 *b*-Quark

After the discovery of the charmed quark, there were two families, and there was a symmetry between leptons (ν_e, e^-; ν_μ, μ^-) and quarks (u, d; c, s). But the situation changed as a new lepton, the τ-lepton, was discovered in July 1975 MARK 1 and SPEAR. Not long after that the fifth quark, the b, was discovered. As in the case of charmed quark, the first evidence for the b-quark was in the observation of a $b\bar{b}$ bound state Upsilon Υ at Fermilab by Herb et al. Some b-flavoured mesons B^0, B^- and baryon Λ_b have been found. The b-quark is expected to decay predominantly to the c-quark.

10.8.1 Solved Problem

$$\Gamma(D^{*+} \to D^+\gamma)/\Gamma(D^{*0} \to D^0\gamma) = \frac{(\mu_c - \mu_{\bar{d}})^2}{(\mu_c - \mu_{\bar{u}})^2}$$

where μ is a quark magnetic moment which we assume to be inversely proportional to the quark mass. Then

$$\frac{\Gamma(D^{*+} \to D^+\gamma)}{\Gamma(D^{*0} \to D^0\gamma)} = \frac{(\mu_c - \mu_{\bar{d}})^2}{(\mu_c - \mu_{\bar{u}})^2} = \frac{\left(\frac{2}{3}\frac{1}{2m_c} - \frac{1}{3}\frac{1}{2m_u}\right)^2}{\left(\frac{2}{3}\frac{1}{2m_c} + \frac{2}{3}\frac{1}{2m_u}\right)^2}$$

$$= \frac{\left(2\frac{m_u}{m_c} - 1\right)^2}{\left(2\frac{m_u}{m_c} + 2\right)^2}$$

Br($D^{*+} \to D^+\gamma$) is small. By accident Br($D^{*+} \to D^0\pi^+$) is about twice as large as Br($D^{*+} \to D^+\pi^0$). The total D^{*0} width is nearly equal to the D^{*+} width.

Note that the quark masses are not real masses and cannot be determined with any real accuracy. We will take here two extreme cases to test the sensitivity of the assumptions:

$$\frac{m_u}{m_c} = \frac{m_\rho}{m_\psi}, \text{ and } \frac{m_u}{m_c} = 0$$

giving

$$\frac{\Gamma(D^{*+} \to D^+\gamma)}{\Gamma(D^{*0} \to D^+\gamma)} = \begin{cases} \dfrac{1}{25} & \text{for } \dfrac{m_u}{m_c} = \dfrac{m_\rho}{m_\psi} \\ \dfrac{1}{4} & \text{for } \dfrac{m_u}{m_c} = 0 \end{cases}$$

11

RECENT DEVELOPMENTS

11.1 Introduction to Grand Unified Theories

The success of the Glashow-Weinberg-Salam theory in reducing the number of fundamental interactions from three to two encouraged attempts to continue the process and develop a theory which tries to unify the electroweak and colour forces. Starting with a rather bold and highly tentative speculation, Grand Unified Theories (GUTs) have blossomed into a vast subject.

We shall first discuss the overview of the subject, we shall then go over to some of the details. We shall, therefore, discuss the introduction, the basic ideas and conclusions and then the details of the group theoretical structure.

As mentioned earlier, all known processes in nature from microscopic to macroscopic can be understood as a manifestation of one or more of the four fundamental interactions: strong, electromagnetic, weak and gravitation, with different strengths and obeying different conservation laws. Unification of all these different forces has been and continues to be a major challenge for physicists. It all began more than a century ago when Maxwell succeeded in unifying the then seemingly different electricity and magnetism. The success in the recent past of (i) Glashow-Weinberg-Salam theory in unifying the weak and electromagnetic interactions (Chapter 9), and (ii) Quantum chromodynamics as a candidate gauge theory of strong interactions has revived this perennial dream of a unified field theory in Physics. It is now believed (at least among the optimists in the area) that, if for the time being we ignore gravity, the synthesis of the strong, electromagnetic, and weak interactions is at hand. Lot of work still needs to be done to convince ourselves, but as we shall attempt to show, the things appear very appealing aesthetically and experimental evidences are awaited to support the grand unified ideas.

11.2 Couplings for the Conventional Interactions

The various interactions are given as follows:

1. Strong SU(3)

$$g_3 \sum_q \bar{q}\gamma^\mu \frac{\lambda_a}{2} q G_\mu^a, \tag{11.1}$$

2. Weak SU(2)

$$g_2 \mathbf{W}_\mu \cdot (\sum_q \bar{q}_L \gamma^\mu \frac{\tau}{2} q_L + \sum_l \bar{l}_L \gamma^\mu \frac{\tau}{2} l_L). \tag{11.2}$$

3. Weak U(1)

$$g_1 B_\mu \sum_{f=q,l} \bar{f} \gamma^\mu (Q_f - T_{3f}) f. \tag{11.3}$$

A clear obstacle to the unification of these interactions is that at low energies the coupling constants g_1, g_2, g_3 are all different, while in a truly unified theory, they must be equal. The resolution of this problem is provided by the renormalization group. The property of asymptotic freedom of strong interaction guarantees that strong interactions get weaker at higher momentum transfer. The renormalization group tells us that $SU(2)$ and $U(1)$ coupling constants change with momenta, as shown in Fig. 11.1: g_1 rising and g_2 intermediate between it and g_3. These evolutions suggest that it is possible that all the three couplings may become equal at some momentum scale at which grand unification can take place.

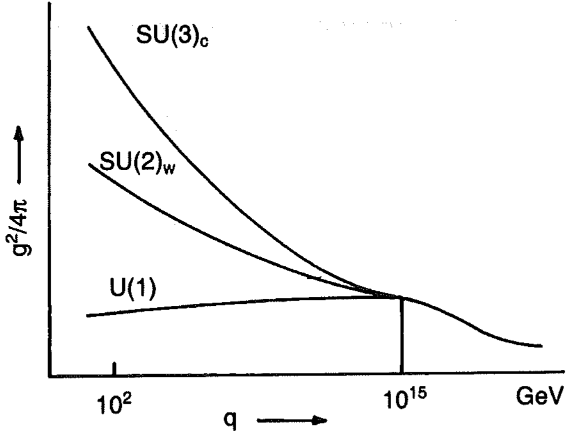

Fig. 11.1 Variation of coupling constants with energy scale.

This is indeed possible but the slow logarithmic variation of different coupling constants means that this occurs at about 10^{15} GeV. While extremely high, this energy is still appreciably lower than the Planck scale, 10^{19} GeV at which quantum gravitational effects become strong and it may seem good to neglect gravity in a first attempt at grand unification of the other fundamental interactions.

11.3 *SU*(5) Model of Georgi and Glashow

Both the QCD and electroweak interactions are expressed in terms of gauge theories. It is, therefore, natural to unify them in a simple group

$$G \supset SU(3)_{\text{colour}} \otimes SU(2)_W \otimes U(1)_Y \tag{11.4}$$

so that there is only one coupling constant. The simplest group is $SU(5)$. The $SU(5)$ model of Georgi and Glashow is the most economical of all the grand unified theories. It is minimal in that only the usual fermions are required; they make up the three generations each of which belongs to a $5 \oplus 10$ representation of $SU(5)$. The $SU(3)_c \otimes SU(2)_W \otimes U(1)_Y$ decomposition of a few low dimensional representations of $SU(5)$ are given now:

SU(5) *representation*	*SU*(2) ⊗ *SU*(3) *representation*
5	(2, 1) + (1, 3)
5*	(2, 1) + (1, 3*)
10	(2, 3) + (1, 3*) + (1, 1)
10*	(2, 3*) + (1, 3) + (1, 1)
24	(1, 8) + (2, 3) + (2, 3*) + (3, 1) + (1, 1)

11.3.1 First Generation of Fermions

There are 15 left-handed states: 2 leptons ν_e, e^-, 1 antilepton e^+, 6 quarks (u_R, u_G, u_B; d_R, d_G, d_B), 6 antiquarks ($\bar{u}_R$, $\bar{u}_G$, $\bar{u}_B$; $\bar{d}_R$, $\bar{d}_G$, $\bar{d}_B$). These 15 fermion states are assigned to the two lowest representations of $SU(5)$ antiquintet and decuplet: $15 = \bar{5} + 10$ in the following way:

$$\bar{5} = (\bar{3}, 1) + (1, 2) = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ \nu_e \end{pmatrix} \tag{11.5}$$

$$10 = \begin{pmatrix} 0 & \bar{u}_3 & \bar{u}_2 & u_1 & d_1 \\ -\bar{u}_3 & 0 & \bar{u}_1 & u_2 & d_2 \\ \bar{u}_2 & -\bar{u}_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix} \tag{11.6}$$

1, 2, 3 stand for colours, red, blue, green. 10 is the antisymmetric part of the product $5 \otimes 5$. We see that the sum of the multiplets $\bar{5}$ and 10 includes just all the 15 fermion states. No extra (new, unknown) states.

The gauge interactions in the $SU(5)$ model take place via exchange of 24-plet of vector bosons. With respect to $SU(3)_c \otimes SU(2)_W$, they transform according to

$$24 = (8, 1) + (1, 3) + (1, 1) + (3, 2) + (3, 2), \tag{11.7}$$

$$24 = \begin{pmatrix} G_1^1 & G_2^1 & G_3^1 & \overline{X}_1 & \overline{Y}_1 \\ G_1^2 & G_2^2 & G_3^2 & \overline{X}_2 & \overline{Y}_2 \\ G_1^3 & G_2^3 & G_3^3 & \overline{X}_3 & \overline{Y}_3 \\ X_1 & X_2 & X_3 & Z^0, \gamma & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & Z^0, \gamma \end{pmatrix} \tag{11.8}$$

with $G_1^1 + G_2^2 + G_3^3 = 0$, corresponding to 8 gluons, $W^\pm$, Z^0, γ, 12 leptoquarks X_i, Y_i.

The gauge group contains extra gauge bosons beyond $W^\pm$, Z^0, γ, and 8 coloured gluons. For $SU(5)$, there are 12 additional gauge bosons, X_i, Y_i called leptoquarks, or diquark bosons, which carry both flavour and colour properties. Leptoquarks are triplets under $SU(3)_c$ and doublets under the electroweak $SU(2)$; hence they are degenerate upto very small $SU(2)$ breaking

$$M_X = M_Y = M_s \tag{11.9}$$

They are presumably very heavy as their effects are not as yet familiar to us. At the unification mass, we require

$$\alpha_1(M_X^2) \approx \alpha_2(M_X^2) \approx \alpha_3(M_X^2) \tag{11.10}$$

where $\alpha_i = g_i^2/4\pi$. From the evolution of the coupling constant, the unification mass and coupling constant turn out to be

$$M_X \approx 10^{15}\,\text{GeV}, \qquad \alpha_i(M_X^2) \approx \frac{1}{40}. \tag{11.11}$$

These bosons and the pattern of symmetry breaking in the $SU(5)$ model are:

$SU(5)$	12 superheavy bosons, $X^{\pm 4/3}$, $Y^{\pm 1/3}$	$M \sim 10^{15}$ GeV.
$SU(3)_c \otimes SU(2) \otimes U(1)$	$W^\pm$, Z^0, γ, gluons	$M = 10^2$ GeV.
$SU(3)_c \otimes U(1)$	8 gluons, photon	all massless.

These extra bosons mediate baryon nonconserving (lepton nonconserving) processes and so can allow for example, the proton to decay to lepton $e^+\pi^0$. The proton is thus not stable. The lifetime of the proton is predicted to be $\tau_p \approx 10^{30}$, which appears to be in conflict with experimental limits $\tau^{\text{exp}}/B(p \to e^+\pi^0) \geq 2 \times 10^{32}$ years. This is a very important consequence of the grand unification.

The interactions of leptoquarks with fermions are shown in Fig. 11.2.

Some of the decay modes of proton decay are represented in Fig. 11.3. Baryon or lepton nonconservation, when combined with hard CP violation, offers a good explanation for the observed matter-antimatter asymmetry of the universe within the framework of big bang cosmology. That is, GUTs provide a viable explanation of the observed ratio

$$(N_B/N_\gamma) \approx 10^{-9},$$

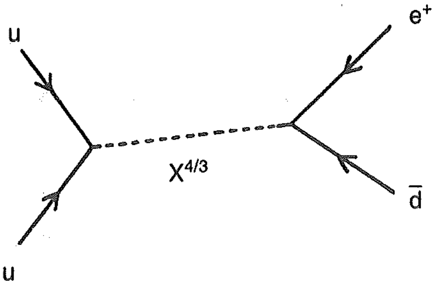

Fig. 11.2 Baryon-lepton violating transition.

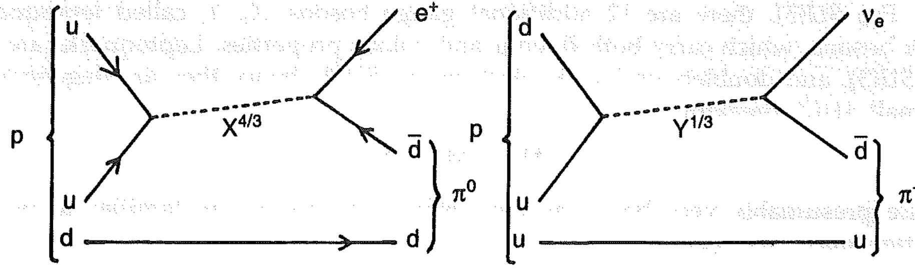

Fig. 11.3 Some proton decay modes.

where N_b is the excess of baryon and N_γ is the background radiation of the universe. This success provides the best available evidence for baryon number nonconservation. It implies that proton should have a finite lifetime.

In summary, some attractive features of the $SU(5)$ model are: It provides a natural explanation for charge quantization, massless neutrino, and observed fermion mass ratio. Furthermore, the theory is truly unified in the sense that the bare couplings associated with the low energy subgroups of $SU(3)_c \otimes SU(2)_W \otimes U(1)_Y$ are constrained to be equal. Other groups which have been considered for grand unification include $SO(10)$ and E_6.

11.4 Beyond the GUT Model

The standard model and the GUTs are incomplete. There are a large number of features unaccounted for, such as the generations of quarks and leptons and many of their properties which have to be put by hand. There is no relationship between the fundamental constituents (fermions) and the quanta (bosons) of forces between them. On the other hand, the quantum theory of gravity is an unrenormalizable theory. A theory which offers an escape from this difficulty at the same time achieving grand unification is a synthesis of general relativity with supersymmetry. Supersymmetry relates bosons and fermions. This is a symmetry which tells us that if we have a boson we must also have a fermion partner and vice-versa. Thus every fermion (quark and lepton) will have a supersymmetric (SUSY) spin zero partner (squark and slepton), while every boson (photon, gluon, W, Z, Higgs) will have a spin 1/2 partner. The quantum of gravitational force, the graviton, which has spin 2, will have a supersymmetric partner, the graviton with spin 3/2. In the symmetric limit the particles and their supersymmetric partners should have equal masses. It is apparent that supersymmetry is a badly broken symmetry as the supersymmetric particles have not been observed, and so they cannot have the same mass as ordinary particles and are probably very heavy. Supersymmetry must then, for the present, be considered as an attractive hypothesis awaiting more data.

11.5 Basics of Supersymmetry

A supersymmetry transformation changes a boson into a fermion and vice-versa. We may write

$$Q_\alpha^\dagger |\text{boson}\rangle = |\text{fermion}\rangle_\alpha, \tag{11.12}$$

$$Q_\alpha |\text{fermion}\rangle = |\text{boson}\rangle_\alpha. \tag{11.13}$$

The generators Q, $Q^\dagger$ are Lorentz spinors obeying the anti-commutation relation

$$\{Q_\alpha^\dagger, Q_\beta\} = p^\mu (\sigma_\mu)_{\alpha\beta}, \tag{11.14}$$

where p^μ is the energy-momentum four vector and σ are the Pauli spin matrices. The above anticommutation relation represents the unusual feature of the supersymmetry: the supersymmetry operators combine to generate translation in space and time. For example, the operation of changing a fermion to a boson and back again to the fermion results in the change of position of the fermion. If supersymmetry is a symmetry of the system, then

$$[Q, H] = 0. \tag{11.15}$$

In the simplest version of supersymmetry, there are four such operators which may be taken as the generators of supersymmetry.

11.5.1 Supersymmetric Quantum Mechanics

We have seen that a free scalar field describes an infinite set of noninteracting harmonic oscillators $a_i^\dagger$, a_i, where i is an index labelling a set. The Hamiltonian of the free field can be written as

$$H_{\text{scalar}} = \sum_i \hbar\omega_i \, (a_i^\dagger a_i + 1/2). \tag{11.16}$$

The ground state of the free scalar quantum field is called vacuum and is described by the conditions

$$a_i |\text{vac}\rangle = 0, \tag{11.17}$$

$$\langle \text{vac} | \text{vac}\rangle = 1. \tag{11.18}$$

The operators $a_i^\dagger$ and a_i create or annihilate, respectively, a single scalar particle of energy $\hbar\,\omega_i$.

Fermionic field

Whereas bosons are represented by scalar field satisfying commutation relations, fermions must be represented by spin-half field obeying anticommutation rules. We define the lowering and raising operators b and $b^\dagger$, respectively, satisfying

$$\{b, b^\dagger\} = 1, \{b, b\} = \{b^\dagger, b^\dagger\} = 0, \tag{11.19}$$

where

$$\{A, B\} \equiv AB + BA. \tag{11.20}$$

If the vacuum is defined through $b|0\rangle = 0$, it is easy to see that

$$b^\dagger|0\rangle = |1\rangle, \qquad b|1\rangle = |0\rangle, \tag{11.21}$$

where $b^\dagger b$ is a number operator.

$$b^\dagger b|1\rangle = |1\rangle, b^\dagger b|0\rangle = 0. \tag{11.22}$$

We may define a Hamiltonian

$$H_{\text{spin}} = \hbar\omega\,(b^\dagger b - 1/2) \tag{11.23}$$

so that the states $|1\rangle$ and $|0\rangle$ have energy equal to $1/2\hbar\omega$ and $-1/2\hbar\omega$, respectively. For an assembly of an infinite number of noninteracting spin 1/2 particles, we may define an infinite set of operators $b_i^\dagger$ and b_i. The vacuum satisfies

$$b_i|\text{vac}\rangle = 0, \tag{11.24}$$

$$\langle\text{vac}|\text{vac}\rangle = 1 \tag{11.25}$$

$b_i^\dagger$ and b_i are identified as creation and annihilation operators, respectively, of a fermionic particle. Since $\{b_i^\dagger, b_i^\dagger\} = 0$, it is possible to create only one particle in the state i. This is the Pauli exclusion principle. Let us now consider a quantum mechanical system that includes the bosonic harmonic oscillator degrees of $a^\dagger$, a and the fermionic spin 1/2 degrees of freedom $b^\dagger$, b. Define

$$Q = a^\dagger b\,(\hbar\omega)^{1/2}, \tag{11.26}$$

$$Q^\dagger = ab^\dagger(\hbar\omega)^{1/2}. \tag{11.27}$$

It is easy to verify that

$$\{Q^\dagger, Q\} = H = H_{\text{osc}} + H_{\text{spin}} = \hbar\omega\,(a^\dagger a + b^\dagger b) = H, \tag{11.28}$$

$$[Q, H] = 0. \tag{11.29}$$

EXERCISE Prove (11.29).

Solution

$$\begin{aligned}\{Q, Q^\dagger\} &= \hbar\omega\,[a^\dagger bab^\dagger + ab^\dagger a^\dagger b] = \hbar\omega\,[a^\dagger abb^\dagger + aa^\dagger b^\dagger b]\\ &= \hbar\omega\,[a^\dagger a(1 - b^\dagger b) + (1 + a^\dagger a)\,b^\dagger b] = \hbar\omega\,(a^\dagger a + b^\dagger b)\\ &= H_{\text{osc}} + H_{\text{spin}} = H.\end{aligned} \tag{11.30}$$

Now,

$$[a^\dagger b, H] = [a^\dagger b, a^\dagger a + b^\dagger b] = [a^\dagger, a^\dagger a]b + a^\dagger[b, b^\dagger b] = -a^\dagger b + a^\dagger b = 0. \qquad (11.31)$$

We see that the anticommuting charges Q and $Q^\dagger$ combine to form the generator of time translation, namely, the Hamiltonian H. The ground state of the system is the state

$$|0\rangle_{\text{osc}}|0\rangle_{\text{spin}} \equiv |0, 0\rangle, \qquad (11.32)$$

where both the oscillator and spin degrees of freedom are in the lowest energy state. This state satisfies

$$Q|0, 0\rangle = Q^\dagger|0, 0\rangle = 0. \qquad (11.33)$$

The excited states have the energy given by

$$E_n = (n + \frac{1}{2})\hbar\omega \pm \frac{1}{2}\hbar\omega, \qquad (11.34)$$

where the sign of the second term depends on the spin state plus corresponding to $|1\rangle$ and the minus to $|0\rangle$. The bosonic state $|n + 1, 0\rangle$ has the same energy as its fermionic partner $|n, 1\rangle$. The values of some lowest energies are given in Table 11.1. It is easy to see that

$$Q|n, 1\rangle = \sqrt{n+1}\,|n + 1, 0\rangle, \qquad (11.35)$$

$$Q^\dagger|n + 1, 0\rangle = \sqrt{n+1}\,|n, 1\rangle, \qquad (11.36)$$

representing the conversion of a fermionic state to a bosonic state and vice-versa. The above example is a simple representation of supersymmetry in quantum mechanics.

Table 11.1 Energy Levels of Bosonic and Fermionic States

Energy	*Boson state*	*Fermion state*
0	$\|0, 0\rangle$	
$\hbar\omega$	$\|1, 0\rangle$	$\|0, 1\rangle$
$2\hbar\omega$	$\|2, 0\rangle$	$\|1, 1\rangle$
$3\hbar\omega$	$\|3, 0\rangle$	$\|2, 1\rangle$

11.6 Super String Theories

Recently, a far more ambitious attempt at unification, which has conceptual similarity with general relativity, has received considerable attention [Green (1985), Schwarz (1985)]. Called the *super string theory*, it proposes to combine the interactions of particle physics with gravity and is essentially a geometric theory. In contrast to conventional field theory, in which the elementary constituents are points in space-time, super string theory postulates that they are curves (or strings) in a certain D-dimensional curved space, where D is considerably larger than 4, with $D = 10$ being the value the current theory requires. The extra dimensions are not directly observable in macroscopic phenomena, however, because the D-dimensional space is supposed to curl up into D-4 compact dimensions, just as ordinary infinite cylinder defines two-dimension space with one compact dimension running around the cylinder and the other non-compact dimension being the cylinder.

11.7 Heavy Flavour-Spin Symmetry

It has not been possible to make predictions on hadronic properties in the low energy nonperturbative regime based on QCD because of our lack of understanding of confinement. Any prediction in this regime comes as a consequence of some symmetry of QCD. We now know that the results of the isospin symmetry are approximately true because the light quark mass difference $m_d - m_u$ is much smaller compared with the QCD scale $\Lambda_{\text{QCD}} \approx$ 300 MeV. The isospin symmetry would be exact in the limit $m_d - m_u \to 0$. $SU(3)$ flavour symmetry is similar, but the corrections due to symmetry breaking here are larger, as $m_s - m_u$ is not small compared with Λ_{QCD}. The chiral symmetry arises because both m_d and m_u are small compared to the QCD scale. If the strange quark mass is treated small compared with the QCD scale then the chiral symmetry group becomes $SU(3)_L \times SU(3)_R$.

Recently, developments have taken place in the theory of heavy quarks which have increased the prospects for the study of non-perturbative QCD in the weak decays of heavy mesons and baryons and for the reliable determination of some of the parameters of the standard model. It has been made possible due to the discovery of a spin-flavour symmetry for heavy quarks which arises in QCD when quark mass is taken to infinity. Certain properties in hadrons containing a heavy quark then become independent of its mass and spin. These ideas have tremendous impact on the phenomenology of heavy hadrons. In particular, these symmetries give rise to restrictive relations among weak decay amplitudes and reduce the number of independent form factors. By relating various matrix elements and fixing normalization of some matrix elements, the heavy quark symmetry has enhanced our predictive ability, allowing in some cases to bypass the difficulties of understanding hadronic structure. Further, with the interplay with phenomenology, this new development may help in the understanding of hadronic structure.

The quark contribution to the QCD Lagrangian density

$$\mathcal{L}_{\text{quark}} = \bar{q}(i\gamma \cdot D - m_q)q + \bar{Q}(i\gamma \cdot D - M_Q)Q \tag{11.37}$$

separates into two pieces: one from the light quarks ($q = u, d, s$) and the other from the heavy quarks ($Q = c, b, t$). Each of these has a distinct symmetry. The light quark sector exhibits an approximate chiral $SU(3)_L \times SU(3)_R$ symmetry as the current quark masses for u, d, and s are very small compared with the typical hadron energy scale. This symmetry is spontaneously broken to the usual $SU(3)_{\text{flavour}}$. Since at the energy scale of heavy hadrons the mass difference of the strange and up (down) constituent quarks is negligible, this $SU(3)$ symmetry is expected to work very well for the heavy hadrons. With this in mind, several authors have used this symmetry to relate different decay amplitudes of charmed and b-hadrons.

On the other hand, in the limit of infinite quark mass, the other side of the scale, the dynamics of a heavy quark in QCD depends only on its velocity and is independent of its mass and spin. As a consequence, two additional symmetries appear in the sector of hadrons containing one heavy quark. Since all the heavy quarks are treated on the same footing, we have an $SU(N)_{\text{HF}}$ (Heavy Flavour) symmetry, N being the number of heavy quarks. The second symmetry is due to the fact that the spin degree of freedom of the heavy quark decouples from the light QCD degree of freedom in this limit. Thus, there is an additional $SU(2)$ spin symmetry corresponding to the spin rotations of the heavy quark moving with a fixed velocity.

For N heavy quarks, therefore, there is an $SU(2N)$ flavour-spin symmetry. This new symmetry is known as the heavy quark symmetry and has seen much progress during the last few years. More precisely, this symmetry ($SU(2N)$) implies that the spectrum of heavy mesons and heavy baryons is independent of the flavour and spin of the heavy quark, so are the transition form factors which describe the weak decays of these heavy hadrons. The ideas of the heavy quark limit are most easily implemented in an effective field theory called the *Heavy Quark Effective Theory (HQET).*

Since the heavy hadrons contain both the heavy and light quarks, both the $SU(3)$ symmetry of the light quarks and the heavy quark $SU(2N)$ symmetry are expected to have implications for the low energy dynamics of heavy hadrons.

A simple result of the spin symmetry of the heavy quark is that 0^--mesons made up of a heavy quark and a light antiquark (e.g. D) and the corresponding vector mesons (e.g. D^*) should be degenerate. Comparing this prediction with the experimental data, we may conclude that the b and c quarks may certainly be treated as heavy and heavy flavour symmetry is thus an $SU(2)$ symmetry denoted by $SU(2)_{\mathrm{HF}}$.

The consequences of the spin-flavour symmetries (referred to as Isgur Wise Symmetry) for the weak semi-leptonic decays of the B-mesons to D and D^* mesons were obtained by Isgur and Wise, Georgi and others who showed that all the form factors are expressible in terms of a single universal function of velocity transfer, which is normalized to unity at zero recoil. The implications of the heavy quark symmetry for the semi-leptonic decays of baryons are that the six real, generally independent form factors that describe the semi-leptonic decays of b-lambda baryons to charmed lambda baryon are expressible in terms of a single universal function of velocity transfer if we assume that b and c quarks are heavy enough to satisfy the requirements of HQET. For the transition in which a heavy quark c is changed to a light quark s, only two form factors are needed. The corresponding theoretical description of decays of heavy hadrons is thus greatly simplified and model dependence is drastically reduced. Further, the reduction in the number of form factors leads to testable experimental predictions. One of the first applications of the ideas of heavy quark symmetry was to semi-leptonic B decays and to extract b to c mixing angle. It was made possible because at zero recoil the matrix element for $B \to D^*$ depends solely on the uncorrected form factor and so could be used to determine V_{cb}. The initial analysis was made by Neubert and gave

$$|V_{cb}| \frac{\tau_B}{1.18\,ps} = 0.045 \pm 0.007. \tag{11.38}$$

In addition, if we assume that the matrix elements for the nonleptonic decays factorize, then the same form factors that determine the properties of semi-leptonic decays also describe the nonleptonic ones. The naive predictions show that HQET performs well when confronted with the experimental data. In particular, one prediction of the HQET within the factorization model is that the asymmetry parameter α for the decay $\Lambda_c \to \Lambda\pi^+$ is -1. Recent experimental results from CLEO and ARGUS seem to confirm this prediction. It is also observed that the decay $\Xi_c^0 \to \Xi^-\pi^+$ may provide a direct test for the factorization hypothesis. However, more stringent tests require better experimental data as also the incorporation of $1/m_c$ effects which have recently been studied upto order $1/M_Q^2$. New data from charm and beauty factories will allow us to test the heavy quark symmetry and to determine reliably some standard model parameters.

11.7.1 The Heavy Quark Approximation

Physical intuition

The central idea is simple. The success of the constituent quark model demonstrates that inside hadrons, strongly bound quarks exchange momentum of magnitude a few hundred MeV. We can think of a typical amount Λ_{QCD} by which the quarks are off-shell in the nucleon as $\Lambda_{QCD} = M_P/3 = 330$ MeV, M_p being the proton mass. In a heavy hadron again the light quarks may be off shell by an amount Λ_{QCD}. However, if the mass M of the heavy quark Q is large, $M_Q >> \Lambda$; then, in fact, this quark is almost on-shell. Moreover, interactions with light quark typically change the momentum by Λ so that the change in the velocity of Q is negligible being of the order of $\Lambda/M_Q << 1$. One can, therefore, think of Q as moving with almost constant velocity and this velocity is, of course, the velocity of the hadron. The velocity of the heavy quark can be modified only by the perturbative processes involving very hard gluons or electro-weak interactions. In the rest frame of the heavy hadron, the heavy quark is practically at rest and acts as a static source of gluons.

An analogy can be drawn from the picture of an atom in the nonrelativistic quantum mechanics. The heavy quark is similar to the atomic nucleus, QCD to electro-magnetic interactions, and the light quark degrees of freedom to the electron; an obvious distinction being that in the hadronic system, the strong interactions of the light degrees of freedom are not calculable because of their nonperturbative nature. The spin of the heavy quark decouples because the (spin-spin) hyperfine magnetic interaction is inversely proportional to the mass of the heavy quark and so is a good quantum number. In the infinite limit, therefore, the low energy hadronic dynamics becomes independent of mass and spin of the heavy quark resulting in an $SU(4)$ symmetry of $b\uparrow$, $b\downarrow$; $c\uparrow$, $c\downarrow$.

The observation that the two new symmetries of flavour and spin of the heavy quark arise in the heavy quark approximation can also be seen more explicitly when we consider the propagator and vertex of the QCD in the infinite mass limit. The heavy quark spinor in the heavy quark approximation satisfies

$$\gamma \cdot v h_v^c(x) = h_v^c(x). \tag{11.39}$$

Consider the usual fermion propagator

$$\frac{1}{\gamma \cdot p - M_Q} \tag{11.40}$$

for heavy quark moving with a velocity v having a residual momentum

$$p^\mu = M_Q v^\mu + k^\mu. \tag{11.41}$$

Looking at the limit $M_Q \rightarrow \infty$, we see that

$$\frac{i}{\gamma \cdot p - M_Q} = \frac{i(\gamma \cdot p + M_Q)}{(p^2 - M_Q)} = \frac{i(M_Q \gamma \cdot v + M_Q + \gamma \cdot k)}{2M_Q v \cdot k + k^2} \tag{11.42}$$

$$= \frac{i(\gamma \cdot v + 1 + \gamma \cdot (k/M_Q)}{(2v \cdot k + k^2/M_Q)} \rightarrow \frac{i}{v \cdot k} \tag{11.43}$$

so that we can take the propagator to be, simply,

$$\frac{i}{v \cdot k}. \tag{11.44}$$

Gluon coupling with a heavy quark is represented by the vertex

$$-igT^{\alpha}\gamma^{\mu}. \tag{11.45}$$

This vertex for the heavy quark is sandwiched between the spinors and so between the on-shell projection operator $\frac{1+\gamma \cdot v}{2}$, giving

$$\frac{1+\gamma \cdot v}{2}\gamma^{\mu}\frac{1+\gamma \cdot v}{2} = v^{\mu} \tag{11.46}$$

so that the gluon vertex is replaced by

$$(-ig)T^{\alpha}v^{\mu}. \tag{11.47}$$

The new Feynman rules have no heavy quark mass dependence and so exhibit explicit symmetry under the change of heavy quark flavour. Also, there are no γ-matrices in the Feynman rules, and hence the heavy quark spin symmetry is also apparent. These rules can be obtained in the heavy quark effective theory with the Lagrangian.

11.7.2 Heavy Quark Effective Theory

The idea of the heavy quark limit can be easily formulated in terms of an effective field theory termed as *Heavy Quark Effective Theory* (HQET). It is useful as it gives a procedure for making explicit calculations.

The heavy quark effective Lagrangian is

$$\mathrm{L} = \bar{h}_v(v^{\mu}D_{\mu})h_v \tag{11.48}$$

which, with the QCD Lagrangian for the gluon and light quarks, describes the structure of hadrons. The heavy quark effective field h_v is a Dirac spinor that satisfies the constraint

$$\gamma \cdot v h_v = h_v$$

or

$$\frac{1+\gamma \cdot v}{2}h_v = h_v. \tag{11.49}$$

It destroys a heavy quark of type i and four-velocity v and is related to the original field operator Q_i by

$$h_v^i(x) = e^{iM_Q v \cdot x}P_+ Q_i(x); \tag{11.50}$$

$P_+(v) = \frac{1}{2}(1 + \gamma \cdot v)$ is an on-shell projection operator. The Lagrangian in (11.48) has the symmetries associated with a quark of infinite mass and is derived from the Dirac Lagrangian

$$\mathcal{L} = \overline{Q}(\gamma \cdot D - M_Q)Q, \tag{11.51}$$

in the infinite mass limit with the help of (11.50). We have done this for the heavy quark. We can do so for the heavy antiquark also. In the infinite mass limit, there is not sufficient energy to create heavy quark-antiquark pairs, so quark and antiquark fields are independent in HQET.

11.7.3 Matrix Elements of Currents between Meson States

A heavy meson consists of a heavy quark Q and a light antiquark q. If we denote by P_i and P_i^* the pseudoscalar (0^-) and vector (1^-) mesons, their quantum numbers are represented in the following interpolating fields

$$P_i(v) = \bar{q}\gamma_5 h_v^i \sqrt{M_P}, \tag{11.52}$$

$$P_i^*(v, \varepsilon) = \bar{q}\gamma \cdot \varepsilon h_v^i \sqrt{M_{P^*}}. \tag{11.53}$$

M_P and M_{P^*} are the masses of the pseudoscalar and vector mesons. The light quark q stands for a column vector in flavour $SU(3)$

$$q = \begin{vmatrix} u \\ d \\ s \end{vmatrix}. \tag{11.54}$$

Thus, each of $P_i(v)$ and $P_i^*(v, \varepsilon)$ is an $SU(3)$ antitriplet. Consider, for example, the matrix element

$$\begin{aligned} \frac{\langle P_j(v')|J_\lambda|P_i(v)\rangle}{\sqrt{M_p M_{p'}}} &= \langle 0|(\bar{q}_{v'}\gamma_5 h_{v'}^j)(\bar{h}_{v'}^j \gamma_\lambda h_v^i)(\bar{h}_v^i \gamma_5 q_v)|0\rangle \\ &= Tr\,[\gamma_5 \frac{\gamma \cdot v' + 1}{2}\gamma_\lambda \frac{\gamma \cdot v + 1}{2}\gamma_5 M], \end{aligned} \tag{11.55}$$

for vector current. We may write

$$M = \langle 0|q_{v'}\ \bar{q}_v|0\rangle = A + B\gamma \cdot v + C\gamma \cdot v' + D\gamma \cdot v\gamma \cdot v' \tag{11.56}$$

from Lorentz invariance. Now, since

$$\gamma \cdot v'\gamma_5 \frac{\gamma \cdot v' + 1}{2} = -\gamma_5 \frac{\gamma \cdot v' + 1}{2}, \tag{11.57}$$

$$\frac{\gamma \cdot v + 1}{2}\gamma_5 \gamma \cdot v = -\frac{\gamma \cdot v + 1}{2}\gamma_5, \tag{11.58}$$

we obtain, from (11.55), the relation

$$\langle P_j(v')|J_\lambda|P_i(v)\rangle / \sqrt{M_P M_{P^*}} = \xi(v \cdot v')\,(v + v')_\lambda, \tag{11.59}$$

where

$$\xi(v \cdot v') = A - B - C + D. \tag{11.60}$$

$\xi(v \cdot v')$ contains the nonperturbative information on the response of the light quark(s) to a change in the velocity of the colour source from v to v'. Further, since the integral of the time component of the vector current is the generator of the heavy flavour group, we have the normalization condition

$$\xi(v \cdot v' = 1) = 1. \tag{11.61}$$

Similarly, we can calculate other marix elements of currents between mesonic states. For the ground state mesons, they turn out to be

$$\left.\begin{aligned}
\langle P_j(v')|J_\lambda|P_i(v)\rangle/\sqrt{M_P M_{P'}} &= C_{ji}\xi(v \cdot v')\,(v + v')_\lambda \\
\langle P_j^*(v')|J_\lambda^v|P_i(v)\rangle/\sqrt{M_P M_{P*}} &= C_{ji}\xi(v \cdot v')\,\varepsilon_{\lambda\mu\nu\sigma}\,\varepsilon'^\mu v'^\nu v^\sigma, \\
\langle P_j^*(v')|J_\lambda^A|P_i(v)\rangle/\sqrt{M_P M_{P*}} &= C_{ji}\xi(v \cdot v')[\varepsilon'_\lambda(1 + v' \cdot v) - v'_\lambda \varepsilon' \cdot v], \\
\langle P_j^*(v')|J_\lambda^V|P_i^*(v)\rangle\sqrt{M_{P*} M_{P*}} &= -C_{ji}\xi(v \cdot v')[\varepsilon' \cdot \varepsilon(v + v')_\lambda - (\varepsilon' \cdot v)\varepsilon_\lambda - (\varepsilon \cdot v')\varepsilon'_\lambda], \\
\langle P_j^*(v')|J_\lambda^A|P_i^*(v)\rangle\sqrt{M_{P*} M_{P*}} &= C_{ji}\xi(v \cdot v')i\varepsilon_{\lambda\mu\nu\sigma}\varepsilon'^\mu \varepsilon^\nu\,(v + v')^\sigma,
\end{aligned}\right\} \tag{11.62}$$

where C_{ji} is a measure of the renormalization effect. The semi-leptonic decays of B mesons require the matrix elements of the vector and axial vector currents between beauty and charm mesons which can be described in terms of fourteen form factors. In general, they are defined by

$$\left.\begin{aligned}
\langle D(v')|V_\lambda|B(v)\rangle/\sqrt{M_B M_D} &= [h_+(v \cdot v')(v + v')_\lambda + h_-(v \cdot v')(v - v')_\lambda], \\
\langle D^*(v')|V_\lambda|B(v)\rangle/\sqrt{M_B M_{D*}} &= h_V(v \cdot v')\,\varepsilon_{\lambda\mu\nu\sigma}\,\varepsilon'^\mu v'^\nu v^\sigma, \\
\langle D^*(v')|A_\lambda|B(v)\rangle/\sqrt{M_B M_{D*}} &= h_{A1}(v \cdot v')\,\varepsilon'_\lambda(1 + v' \cdot v) - h_{A2}(v \cdot v')v'_\lambda \varepsilon' \cdot v, \\
&\quad - h_{A3}(v \cdot v')\varepsilon'^* \cdot v v'_\lambda, \\
\langle D^*(v')|V_\lambda|B^*(v)\rangle/\sqrt{M_{B*} M_{D*}} &= -\varepsilon^{*\prime} \cdot \varepsilon[h_1(v \cdot v')(v + v')_\lambda + h_2(v \cdot v')(v - v')_\lambda, \\
&\quad + h_3(v \cdot v')(\varepsilon^{*\prime} \cdot v')\varepsilon_\lambda - h_4(v \cdot v')(\varepsilon \cdot v')\varepsilon^{*\prime}_\lambda, \\
&\quad - \varepsilon \cdot v' \varepsilon^* \cdot v\{h_5(v \cdot v')v_\lambda + h_6(v \cdot v')v'_\lambda\}], \\
\langle D^*(v')|A_\lambda|B^*(v)\rangle/\sqrt{M_{B*} M_{D*}} &= i\varepsilon_{\lambda\mu\nu\sigma}\varepsilon'^\mu \varepsilon^\nu[h_7(v \cdot v')(v + v')^\sigma + h_8(v \cdot v')(v - v')^\sigma
\end{aligned}\right\} \tag{11.63}$$

At leading order in the heavy quark expansion, therefore,

$$h_+ = h_v = h_{A1} = h_{A3} = h_1 = h_3 = h_4 = h_7 = \xi \tag{11.64}$$

while the remaining six form factors vanish. Using these relations, predictions can be made on the decay rates and asymmetry parameters. Since in an actual decay, the Isgur-

Wise function is needed at points away from the zero recoil, it is parametrized in a variety of forms.

11.7.4 Baryon Wave Functions

Similar ideas can also be applied to the case of baryons. A heavy baryon contains a heavy quark and two light quarks, which we refer to as diquark. The two quarks (or a diquark) are in a symmetric sextet or antisymmetric antitriplet state of $SU(3)$. Since heavy quark is a singlet in $SU(3)$, the transformation property of a heavy baryon is the same as that of light diquark it contains. Spin 1/2 baryons in the antitriplet state are

$$\Lambda_Q = 1/\sqrt{2}(ud - du)Q, \tag{11.65}$$

$$\Xi_Q^{+} = 1/\sqrt{2}(us - su)Q, \tag{11.66}$$

isospin up state, and

$$\Xi_Q^{-} = 1/\sqrt{2}(ds - sd)Q, \tag{11.67}$$

isospin down state.

The light quarks in these are in isospin zero and spin zero state. Spin 1/2 baryons in the sextet state are

$$\Sigma_Q^{+1} = uuQ, \quad \Sigma_Q^{0} = 1/\sqrt{2}(ud + du)Q, \quad \Sigma_Q^{-1} = ddQ,$$

$$\Xi'^{+} = 1/\sqrt{2}(us + su)Q,$$

isospin up state

$$\Xi'^{-} = 1/\sqrt{2}(ds + sd)Q,$$

isospin down state

$$\Omega_Q = ssQ. \tag{11.68}$$

The light diquark in these baryons has isospin one and spin one.

Interpolating fields in terms of diquark fields of the light quarks

Since the ground state baryons have even parity, the diquark must also have even parity. So the diquark in $B_{\bar{3}}$ has spin parity 0^+ and, therefore, it is represented by a Lorentz scalar field ϕ. Similarly, the diquark in B_6 multiplet has spin parity 1^+. It is represented by an axial vector field ϕ_μ. The interpolating fields are

$$B_{\bar{3}}(v, s) = \bar{u}(v, s)\phi_v h_v, \tag{11.69}$$

$$B_6(v, s) = \bar{B}_\mu(v, s, k)\phi_v^\mu h_v; \; k = 1, 2, \tag{11.70}$$

where $k = 1$ for spin 1/2 baryon, $k = 2$ for spin 3/2 baryon. The wave functions, as given by Georgi [5], are

$$\bar{B}_\mu(v, s, k = 1) = 1/\sqrt{3}\,\bar{u}(v, s)\gamma_5(\gamma_\mu + v_\mu), \tag{11.71}$$

$$\bar{B}_\mu(v, s, k = 2) = \bar{u}_\mu(v, s). \tag{11.72}$$

$u_\mu(v, s)$ and $u(v, s)$ are the Rarita-Schwinger vector spinor and the usual Dirac spinor, respectively. [The factor $1/\sqrt{3}$ comes from normalization. To prove that, let us remind ourselves that B_μ is a pseudovector and so can be written as

$$B_\mu(v, s, k = 1) = (A\gamma_\mu + Bv_\mu)\gamma_5 u(v, s). \tag{11.73}$$

This must satisfy $v^\mu B_\mu = 0$, yielding thereby

$$(-A + B)\gamma_5\, u(v, s) = 0$$

so that $A = B$. Now A can be found by the normalization $\bar{B}^\mu B_\mu = -1$. Taking the Hermitian conjugate of (11.73), we get

$$B_\mu^\dagger = A^* u^\dagger \gamma_5 \gamma^0 (\gamma_\mu + v_\mu)\gamma^0$$

so that

$$\begin{aligned}\bar{B}_\mu = B_\mu^\dagger \gamma^0 = -A^* \bar{u}\gamma_5(\gamma_\mu + v_\mu) &\Rightarrow |A|^2 \bar{u}\gamma_5(\gamma^\mu + v^\mu)\,(\gamma_\mu + v_\mu)\,\gamma_5 u(v, s)\\ &= -|A|^2 \bar{u}(-\gamma^\mu + v^\mu)(-\gamma_\mu + v_\mu)\,u(v, s) = -3|A|^2 = -1\end{aligned}$$

giving $|A| = 1/\sqrt{3}$. The negative sign is taken by convention. The corresponding interpolating fields for the antibaryons are

$$\left.\begin{aligned}\bar{B}_3(v, s) &= \bar{h}_v \phi_v^\dagger u(v, s),\\ \bar{B}_6(v, s) &= \bar{h}_v \phi_v^{\dagger\mu} B_\mu(v, s, k);\ k = 1, 2.\end{aligned}\right\} \tag{11.74}$$

Here again, $k = 1$ for spin 1/2 baryon, $k = 2$ for spin 3/2 baryon, with,

$$\left.\begin{aligned}B_\mu(v, s, k = 1) &= -\frac{1}{\sqrt{3}}(\gamma_\mu + v_\mu)u(v, s),\\ B_\mu(v, s, k = 2) &= u_\mu(v, s).\end{aligned}\right\} \tag{11.75}$$

The minus sign is the result of the anticommutation relation

$$\{\gamma_5, \gamma_0\} = 0.$$

Further,

$$\left.\begin{aligned}\gamma \cdot v B_\mu(v, s) &= B_\mu(v, s),\\ v^\mu B_\mu(v, s) &= 0.\end{aligned}\right\} \tag{11.76}$$

Normalizing, we get

$$\bar{u}(v, s')u(v, s) = \delta_{ss'}, \qquad \bar{u}_\mu(v, s')u_\mu(v, s) = -\delta_{ss'}. \tag{11.77}$$

11.7.5 The Matrix Elements of Currents between Baryon States

For the matrix elements of the vector and axial vector between heavy baryon states, we may write

$$\begin{aligned}
\langle \Lambda_{Qj}(v', s')|J_V^{ji}|\Lambda_{Qi}(v, s)\rangle &= \langle 0|\overline{u}(v', s')\phi_{v'}h_{v'}\overline{h}_{v'}\gamma_\lambda h_v\overline{h}_v\phi_v^\dagger u(v, s)|0\rangle \\
&= \langle 0|\overline{u}(v', s')h_{v'}\overline{h}_{v'}\gamma_\lambda h_v\overline{h}_v u(v, s)|0\rangle \langle 0|\phi_{v'}\phi_v^\dagger|0\rangle \\
&= \overline{u}(v', s')\gamma_\lambda u(v, s)\,\langle 0|\phi_{v'}\phi_v^\dagger|0\rangle, \qquad (11.78) \\
\langle \Lambda_{Qj}(v', s')|J_A^{ji}|\Lambda_{Qi}(v, s)\rangle &= \langle 0|\overline{u}(v', s')\phi_{v'}h_{v'}\overline{h}_{v'}\gamma_\lambda\gamma_5 h_v\overline{h}_v u(v, s)\phi_v^\dagger|0\rangle \\
&= \langle 0|\overline{u}(v', s')h_{v'}\overline{h}_{v'}\gamma_\lambda\gamma_5 h_v\overline{h}_v u(v, s)|0\rangle \langle 0|\phi_{v'}\phi_v^\dagger|0\rangle \\
&= \overline{u}(v', s')\gamma_\lambda\gamma_5 u(v, s)\,\langle 0|\phi_{v'}\phi_v^\dagger|0\rangle. \qquad (11.79)
\end{aligned}$$

We parametrize $\langle 0|\phi_{v'}\phi_{v'}^\dagger|0\rangle$ as before and write

$$\langle 0|\phi_{v'}\phi_{v'}^\dagger|0\rangle = A + B\gamma\cdot v + C\gamma\cdot v' + D\gamma\cdot v\gamma\cdot v',$$

a Dirac matrix to be inserted in (11.78). In the limit of infinite mass, v is nearly equal to v' and so $\gamma\cdot v$ and $\gamma\cdot v'$ can be replaced by 1, giving a single form factor as a function of a Lorentz invariant velocity transfer

$$\omega = v\cdot v' = \frac{M_{\Lambda Qj}^2 + M_{\Lambda Qi}^2 - q^2}{2M_{\Lambda Qj}M_{\Lambda Qi}}. \qquad (11.80)$$

So,

$$\langle \Lambda_{Qj}(v', s')|J_V^{ji}|\Lambda_{Qi}(v, s)\rangle = A(\omega = v\cdot v')\overline{u}(v', s')\,\gamma_\lambda\, u(v, s),$$

$$\langle \Lambda_{Qj}(v', s')|J_A^{ji}|\Lambda_{Qi}(v, s)\rangle = A(\omega = v\cdot v')\overline{u}(v', s')\,\gamma_\lambda\gamma_5 u(v, s).$$

In general, the semi-leptonic decays are described by the six form factors defined by

$$\begin{aligned}
\langle \Lambda_{Qj}(v', s')|\,J_{v-A}^{ji}\,|\Lambda_{Qi}(v, s)\rangle = \overline{u}_{Qj}(v')[\{F_1(v\cdot v')\gamma_\mu + F_2(v\cdot v')\,v_\mu + F_3(v\cdot v')v'_\mu\} \\
- \{G_1(v\cdot v')\gamma_\mu + G_2(v\cdot v')\,v_\mu + G_3(v\cdot v')v'\}\gamma_5]u_{Qi}.
\end{aligned}$$

In the leading order the results from HQET are

$$\left.\begin{aligned}
F_1(v\cdot v') &= G_1(v\cdot v'), \\
F_2(v\cdot v') &= G_2(v\cdot v') = F_3(v\cdot v') = G_3(v\cdot v') = 0.
\end{aligned}\right\} \qquad (11.81)$$

11.7.6 Semi-leptonic Baryon Decays

Heavy to heavy transition

As an example of the method, we choose the semi-leptonic decays of the bottom baryon Λ_B, because it is simple.

$$\Lambda_b^0 \to \Lambda_c^+ e^- \overline{\nu}_e.$$

The matrix element is

$$\langle \Lambda_c; v' |\, J_\mu^{\text{weak}} \,| \Lambda_b; v\rangle = \bar{u}(v')\gamma_\mu(1-\gamma_5)u(v)F_\Lambda(\omega). \tag{11.82}$$

Further, this universal form factor $F(\omega)$ is identical to the elastic form factor of Λ_b and is normalized to 1 at $\omega = 1$.

Heavy to light transition

In heavy to light transition such as

$$\Lambda_b^0 \rightarrow \Lambda,$$

we do not have such a large symmetry. However, the symmetry in the heavy baryon wave function does reduce the number of form factors even in this case. The matrix element of weak current for a heavy baryon Λ_c into a light baryon B is given by

$$\langle B(p') |\, \bar{q} L_\mu h \,| \Lambda_c\rangle = u(p')[A_1(p'\cdot v) + \gamma\cdot v A_2(p'\cdot v)]L_\mu u_{\Lambda c}(v). \tag{11.83}$$

Here, there are two independent form factors. $L_\mu = \gamma_\mu(1-\gamma_5)$ is the left-handed current operator, $v = p/M$ is the four velocity of Λ_c and p' is the four momentum of the daughter baryon B.

Wisgur symmetry

We would like to show the application of the simple idea of the conservation of the spin of the heavy quark. Let us first look at the derivation of an important commutator which we have derived with the method we have already used in earlier chapters. We want to prove that

$$[S_{Qj}^3, V_3^{ji}] = -\frac{1}{2}A_0^{ji},$$

where

$$V_3^{ji} = \bar{Q}_j \gamma_3 Q_i$$

$$S_{Qj}^3 = \int :\bar{Q}_j^\dagger \frac{1}{2} i\gamma^1\gamma^2 Q_j : d^3x = \int :\bar{Q}_j^\dagger [\gamma^1, \gamma^2] Q_j : d^3x.$$

We use the identity

$$[\psi_a^\dagger O_1 \psi_b, \psi_c^\dagger O_2 \psi_d] = \delta_{bc}\psi_a^\dagger O_1 O_2 \psi_d - \delta_{ad}\psi_c^\dagger O_2 O_1 \psi_b.$$

The identity $a = j$, $b = j$, $c = j$ and $d = i$, and hence only the first term in the identity is nonzero, and gives us

$$[S_{Qj}^3, V_3^{ji}] = \left[\int : Q_j^\dagger \frac{1}{2} i\gamma^1\gamma^2 Q_i : d^3x, \bar{Q}_j\gamma_3 Q_i\right]$$

$$= Q_j^\dagger \frac{1}{2}(i\gamma^1\gamma^2)(\gamma^0\gamma^3)Q_i = \frac{1}{2}Q_j^\dagger\gamma_5 Q_i = \frac{1}{2}A_0^{ji}.$$

11.7.7 Matrix Elements of Currents between Baryon States

We may write

$$\langle \Lambda_{Qj}(v', s')|J^{ji}_{V,A}|\Lambda_{Qi}(v, s) = \bar{u}(v', s')\,\Gamma^{ji}_{V,A}u(v, s),$$

where

$$\Gamma^{ji}_{V} = F^{ji}_{1}\gamma_{\nu} + F^{ji}_{2}v_{\nu} + F^{ji}_{3}v'_{\nu},$$

$$\Gamma^{ji}_{A} = G^{ji}_{1}\gamma_{\nu}\gamma_{5} + G^{ji}_{2}v_{\nu}\gamma_{5} + G^{ji}_{3}v'_{\nu}\gamma_{5}.$$

The form factors are functions of $\omega = v \cdot v'$ defined earlier. In the heavy quark limit,

$$\gamma \cdot vQ_i = Q_i, \qquad \overline{Q}_j\gamma \cdot v' = \overline{Q}_j,$$

and so

$$(v - v')^{\nu}V^{ji}_{\nu} = 0, \quad (v + v')^{\nu}A^{ji}_{\nu} = 0,$$

giving

$$(v - v')^{\nu}\bar{u}(v', s')\Gamma^{ji}_{V,\nu}u(v, s) = 0, \quad (v + v')^{\nu}\bar{u}(v', s')\Gamma^{ji}_{A,\nu}u(v, s) = 0.$$

Applying these constraints, we obtain

$$(F_2 - F_3)\,(1 - v \cdot v') = 0, \qquad \Rightarrow F_2 = F_3.$$

(v and v' being arbitrary). Similarly, we can prove that

$$G_2 = -\,G_3.$$

The matrix elements of the vector and axial vector can be related through the commutation relation giving

$$F_1 = -\,G_1, \qquad F_2 = F_3 = G_2 = G_3 = 0$$

through the relations

$$\langle \Lambda_{Qj}(0, \tfrac{1}{2})|V^{ji}_{3}|\Lambda_{Qi}(0, \tfrac{1}{2})\rangle = -\,\langle \Lambda_{Qj}(0, \tfrac{1}{2})|A^{ji}_{0}|\Lambda_{Qi}(0, \tfrac{1}{2})\rangle$$

$$S^{3}_{Q}|\Lambda_{Q}(0, \tfrac{1}{2})\rangle = \tfrac{1}{2}|\Lambda_{Q}(0, \tfrac{1}{2})\rangle.$$

11.7.8 Nonleptonic Decays

To lowest order in the weak interaction, the nonleptonic weak interaction Hamiltonian H in the standard model can effectively be written in the current × current form

$$H = \frac{G_F}{\sqrt{2}}\,J_{\mu}\,J^{\mu} + h.c.,$$

where the weak current J is given by

$$J = (\bar{u}\bar{c}\bar{t})\gamma_\mu(1-\gamma_5)\begin{vmatrix} d' \\ s' \\ b' \end{vmatrix}.$$

The weak eigenstates d', s', and b' are related to the mass eigenstates d, s, and b through the Cabibbo-Kobayashi-Maskawa mixing matrix.

11.7.9 The Factorization Hypothesis

In this approach for nonleptonic weak decays, one assumes that the long distance QCD effects occur mainly in the initial and final state configurations. This is referred to as the factorization hypothesis as it allows the decay amplitude to be written as a product of matrix elements of currents, e.g.

$$A(D^+ \to \bar{K}^0\pi^+) = [a_1\langle\bar{K}^0|J^\mu|D^+\rangle\langle\pi^+|j_\mu|0\rangle + a_2\langle\bar{K}^0|j^\mu|0\rangle\langle\pi^+|J_\mu|D^+\rangle].$$

This is similar to the manner in which semi-leptonic decay amplitudes are constructed where factorization occurs automatically since no strong coupling occurs between the leptons and quarks. Although this hypothesis, so far, lacks a good theoretical justification, yet it seems to work phenomenologically for the decays of heavy mesons. The nonleptonic decays of heavy mesons and heavy baryons have been analysed in the heavy quark approximation with the help of factorization hypothesis. One of the results of these considerations for the heavy baryon decays is that the asymmetry parameter comes out to be –1, which appears to be in agreement with recent experimental result.

11.7.10 Nonleptonic Decays of Heavy Baryons

The factorized amplitude for the decay $\Lambda_c \to BX$, where X may be a pseudoscalar meson. Or, a vector meson can be written as

$$M_{\Lambda_c \to BX} = \frac{G_F}{\sqrt{2}} V_{cj}\, V^*_{kl} H_\mu X^\mu \tag{11.84}$$

with the appropriate CKM matrix elements, where

$$X^\mu = \langle X|J^\mu|0\rangle = f_P q^\mu, \tag{11.85}$$

with X as the pseudoscalar meson P, q as its fourth momentum, and f_p its decay constant. H_μ is defined by

$$H_\mu = \langle B(p')|\,\bar{q}L_\mu h\,|\Lambda_c\rangle = \bar{u}_B(p')[A_1(p'\cdot v) + \gamma\cdot vA_2(p'\cdot v)]L_\mu u_{\Lambda_c}(v). \tag{11.86}$$

A_1 and A_2 are two real form factors describing the weak current for heavy to light baryon transition. A_1 and A_2, in general, depend on the flavour of the light baryon but not on X. Thus all decays of the type $\Lambda_c \to B$ are related to each other. Substituting

$$q = m_{\Lambda_c}v - p',$$

we obtain

$$M_{\Lambda_c \to BX} = \frac{G_F}{\sqrt{2}} V_{cj} V_{kl}^{*} \bar{u}_B(p') \left[g_1 - g_2\gamma_5\right] u_{\Lambda_c}(v). \tag{11.87}$$

g_1 and g_2 are given by

$$g_1 = A_1(m_{\Lambda_c} - m_\Lambda) + A_2(m_{\Lambda_c} + m_\Lambda - 2E),$$

$$g_2 = A_1(m_{\Lambda_c} + m_\Lambda) - A_2(m_{\Lambda_c} - m_\Lambda - 2E).$$

In the limit of negligible pion mass,

$$\frac{g_1}{g_2} = \frac{(m_{\Lambda_c} - m_\Lambda)}{(m_{\Lambda_c} + m_\Lambda)} = 0.34.$$

This gives the value of α, the decay asymmetry parameter to be –1, which is very close to the measured value.

11.7.11 Corrections

Heavy-quark symmetry makes many predictions in the infinite mass limit. However, as we want to apply these ideas to c and b quarks which are not really infinitely massive, we must include $1/m$ corrections. Fortunately, the effective field theory allows a systematic inclusion of corrections, and hence there is some predictive power from the symmetry even when it is broken.

11.7.12 Conclusions

It appears that it may be useful to study heavy hadrons in the heavy quark approximation. There are many advantages of this limit: heavy quark acts as a pointlike probe of the light constituent quarks, and the relativistic effects are simplified. The results of the symmetries of the heavy quark limit may be employed to act as consistency checks on models. HQET, coupled with factorization hypothesis, can be used to study nonleptonic decays of heavy hadrons. The naive predictions on charm baryon decays show that HQET performs well when confronted with the experimental data. It is also observed that the decay $\Xi_c^0 \to \Xi^- \pi^+$ may provide a direct test for the factorization hypothesis. However, more stringent tests require better experimental data as also the incorporation of $1/m$ effects. $1/m$ and other corrections have already been studied systematically upto order $1/m$. Eventually, a more detailed study of the heavy-to-light form factors is called for. For the knowledge of the universal form factors (Isgur-Wise function), we need nonperturbative QCD, but an understanding of these functions is ultimately necessary for any quantitative analysis based on heavy quark symmetry. In fact, here the interplay of HQET with models may help in the understanding of some aspects of confinement. In future, it may be possible to compute the functions from the first principle, by using the effective theory on the lattice. In the mean time, QCD sum rules have been recently been employed. New data from charm and beauty factories will allow us to test heavy quark symmetry and to determine reliably some standard model parameters. Heavy quarks may well turn out to be a useful tool to understand the nature of QCD of the light quarks.

11.8 Cosmology and Particle Physics

Three areas of recent activity at the interface of particle physics and cosmology are:

1. Nucleosynthesis and neutrino counting
2. The dark matter problem
3. The formation of galaxies and large scale structure.

Both particle physics and cosmology have their standard models which are getting confirmed to a reasonable degree of accuracy by the present generation of experiments. The particle model is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, and the cosmological model is the hot big bang. While the Hubble's work in the 1920's established the expanding universe, the two key quantitative observations

1. the microwave background, and
2. the Big Bang Nucleosynthesis (BBN) and light element abundances

establish the modern physical cosmology and the hot big bang model. The second of these observations is more directly connected to high energy physics. Just as the new COBE results have given renewed confidence in the 3 K background argument, the LEP collider has given us renewed confidence in the BBN argument.

Note that the microwave background probes events at temperatures of the order of 10^4 K and time of the order of 10^5 years, whereas the light abundances probe the universe at temperatures of the order of 10^{10} K and times of the order of 1 s. Thus it is the nucleosynthesis results that have played the most significant role in leading to the particle physics-cosmology merger, that has taken place in this past decade or so.

11.9 Suggested Further Reading*

Bjorken, J. (1990): SLAC preprint SLAB-PUB 5278.

Bjorken, J.D. (1990): SLAC-Report 378, *in* Proc. 18th SLAC Summer Institute of Particle Physics, Hawthorne (Ed.).

Georgi, H. (1991): Lectures at TASI, HUTP-91-A039.

Grinstein, B. (1991): HUTP-91-A040, SSCL-Preprint 17.

Isgur, N. and Wise, M.B. (1989): *Phys. Lett.*, **B232**, 113; (1990): **B237**, 527.

There are now several lecture notes which discuss in detail the ideas, calculational methods and results of the new (so-called Isgur-Wise) symmetry. They are:

Mannel, T. Heavy quark effective theory, IKDA 92/6, Jan. 1992. Inst. Kernphysik, Tech. Hochschule, Darmstadt.

Neubert, M. (1994): SLAC report No. SLAC-PUB-6263, *Phys. Rept.*, **245**, 259.

Wise, M.B. (1991): CALT-68-1721, Lectures at Lake Louise Winter Institute.

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*These references relate to heavy quark effective theory.

Appendix A

ELEMENTS OF GROUP THEORY

A.1 Group

A set $a, b, c, \ldots$ of elements, together with an operation called group multiplication, is said to form a group if the following postulates are satisfied:

(i) ***Closure.*** The product ab of any two elements a and b of the group is unique and belongs to the group. That is, if

$$a \subset G,\ b \subset G, \quad \text{then} \quad ab \subset G. \tag{A.1}$$

(ii) ***Associativity.*** In a product of three or more elements of the group, the order of multiplication is immaterial, i.e.

$$a(bc) = (ab)c = abc. \tag{A.2}$$

(iii) ***Identity.*** The set contains an element e (or 1), known as the *identity element* such that for any element a,

$$ae = ea = a. \tag{A.3}$$

(iv) ***Inverse.*** For any element of the group there exists an *inverse* a^{-1} such that

$$a^{-1}a = aa^{-1} = e. \tag{A.4}$$

The simplest group is the trivial group consisting of only one element, the identity e. Other examples are given now.

Example 1: The real numbers form a group under the operation of addition. So do complex numbers. In both the cases, 0 is the identity element.

Example 2: The set of real numbers excluding zero form a group under multiplication. So do the complex numbers. The identity in both the groups is 1.

Example 3: The set of all $n \times n$ unitary matrices form a group under matrix multiplication.

The definition of the group does not require that the multiplication be commutative. However, a group for which

$$ab = ba \tag{A.5}$$

for all elements is said to be Abelian (or commutative).

A group is finite if it has a finite number of elements. Otherwise, it is an infinite group. The number of elements of a finite group is called the *order of the group.*

A.1.1 Subgroup

A subgroup of G is a subset of elements which satisfy the group postulates. The group G itself and the group consisting of unit element are trivial subgroups of G.

A subgroup S is invariant if for every element s, gsg^{-1} is also in S, for any element g in G. In an invariant subgroup each of its elements commutes with each element of the complete group. Thus two-dimensional rotations about a fixed axis form an invariant subgroup of rotations in three dimensions.

A.1.2 Simple Group

A group is said to be simple if it contains no invariant continuous subgroups except the unit element. A group is *semi-simple* if it has no Abelian invariant subgroups.

A.2 Representation of a Group

If to each element a of the group G, there is associated an operator $T(a)$ such that

$$T(a)T(b) = T(c) \quad \text{if } ab = c, \tag{A.6}$$

$$T(a^{-1}) = T^{-1}(a) \quad T(e) = 1 \text{ the unit element}, \tag{A.7}$$

the operators $T(a)$ and the basic vectors on which they act form the representation of the group and the basis of the representation respectively. If $T(a)$ is represented by a matrix, we have a matrix representation of the group. The dimensionality of the matrices is the dimension of the representation.

A.2.1 Reducible Representation

A representation is completely reducible if every matrix of the representation can be expressed in the block diagonal form

$$\begin{bmatrix} a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & h & i & j & 0 & 0 & 0 & 0 \\ 0 & 0 & k & l & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n & o & p & q \\ 0 & 0 & 0 & 0 & 0 & r & s & t & u \\ 0 & 0 & 0 & 0 & 0 & v & w & x & y \\ 0 & 0 & 0 & 0 & 0 & z & \alpha & \beta & \gamma \end{bmatrix} \tag{A.8}$$

since each submatrix forms a representation.

A.2.2 Irreducible Representation

If such a reduction is not possible, the representation is known as irreducible representation. One of the most important properties of group is the dimensionality of its irreducible representations. For example, for three-dimensional rotations, the dimensionality of the irreducible representations is $2j + 1$, where j is an integer or half-odd integer.

The representation is *unitary* if the operators $T(a)$ as also the corresponding matrices are unitary. Two representations are said to be *equivalent* if one can be obtained from the other by a change of basis, i.e.

$$T'(a) = AT(a)A^{-1} \tag{A.9}$$

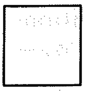
The most interesting groups in particle physics are the continuous groups, which depend on a finite number of parameters. They are known as *Lie groups.*

We will, in particular, be interested in compact Lie groups. The term 'compact' refers to a global property of the group which we will not discuss here.

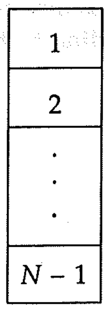
The theory of group representations is of particular importance to particle physics because the structure of the irreducible representations of the group that gives the symmetries of the system determines the degeneracy of its energy levels. The degree of degeneracy of a given state equals the dimension of some irreducible representation of the symmetry group. This property is very useful as it enables us to identify the group of symmetries that cause a given degeneracy by matching the observed degeneracies to the known irreducible representations of the group.

A.3 Young Tableau for Unitary Groups

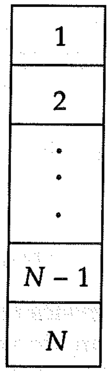
The calculations of the dimensionality of an irreducible representation of $SU(n)$ can be made by an elegant and fast diagrammatic technique, that of the Young tableau. This method is particularly useful for representations of $SU(n)$, where n is larger than 3 or for the higher representations of $SU(3)$. The Young tableau are also fun to play with. Suppose that we are interested in the irreducible representations of $SU(n)$. The fundamental representation (a one-particle state) will be denoted by a box



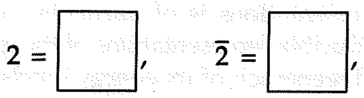
while a column of $N - 1$ boxes denotes the conjugate representation:



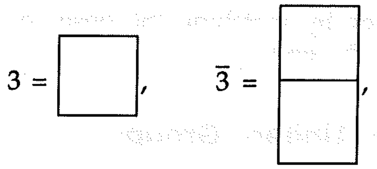
A column of N boxes is a singlet.



Hence in $SU(2)$,

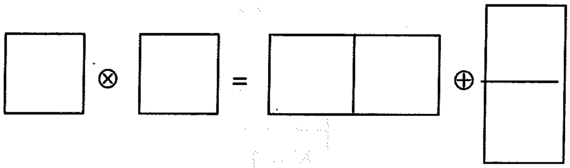


but in $SU(3)$ we have



and hence the fundamental and conjugate representations are the same in $SU(2)$ as shown in Section 4.3.1, but different in $SU(3)$ and higher unitary groups. The Young diagram with only one row is associated with totally symmetric representation and that with only one column to a totally antisymmetric one. All other diagrams represent mixed symmetry.

Now suppose that we want to obtain the irreducible representations in the direct product of two fundamental representations. They will correspond to two particle states. We start with the corresponding Young tableaux, a box for each one of them. We can add the second box corresponding to the second fundamental representation to the first one, either to make a row or a column. Conventionally, these are used to represent two particle symmetric and antisymmetric states respectively. We give below the dimensions of representations corresponding to the row or column for $SU(2)$, $SU(3)$ and $SU(4)$.



(A.10)

$$\begin{array}{lccccccc} SU(2) & 2 & \otimes & 2 & = & 3 & \oplus & 1, \\ SU(3) & 3 & \otimes & 3 & = & 6 & \oplus & \bar{3}, \\ SU(4) & 4 & \otimes & 4 & = & 10 & \oplus & 6, \\ & & & \vdots & & & & \\ SU(N) & N & \otimes & N & = & \frac{1}{2}N(N+1) & \oplus & \frac{1}{2}N(N-1). \end{array}$$

In general, the product of two fundamental representations has the above row or column structure.

For a number of particles, one can join boxes corresponding to each particle in many different ways to get different representations. The Young tableaux with only one row are associated with a totally symmetric representation.

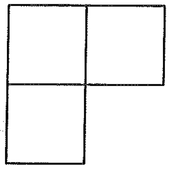
How do we deduce the dimension of an arbitrary $SU(N)$ representation? The calculation of an arbitrary Young tableau involves the ratio of two numbers. We calculate the numerator and denominator as follows:

Calculation of the numerator: For any diagram, put N in each of the diagonal boxes starting from the top left-hand corner. In the diagonal immediately above and below, put $N - 1$ and $N + 1$ respectively. In the next diagonals, put $N - 2$ and $N + 2$ and so on. For example, the numerator will be the product of all these numbers.

Calculation of the denominator: The denominator is given by the product of the "hooks". The hook corresponding to each box is the number of squares we must pass through entering along the appropriate row from the right and leaving the diagram from the bottom of the appropriate column.

N	$N+1$	$N+2$	$N+3$	$N+4$	$N+5$
$N-1$	N	$N+1$			
$N-2$					
$N-3$					
$N-4$					

There is a value of the hook associated with each box. To find this value, do the following: Draw a line entering the right-hand end of the row in which the box in question lies. On entering this box, turn this line downwards through 90° and proceed down the column until the line leaves the diagram. The total number of boxes that the line has passed through, including the box in question, is the value of the hook associated with that box. The product of all the hooks is the denominator. For example, in $SU(3)$ consider the diagram



Numerator = $3 \times 4 \times 2 = 24$,

Denominator = $3 \times 1 \times 1 = 3$,

Dimension = $24/3 = 8$.

Product of two representations (Product of two tableaux)

The decomposition into irreducible representations of the product of two representations can be easily obtained with the help of Young diagrams. The Young tableau corresponding to one (say, the first) of the two representations is taken as fixed. One must add to this fixed tableau, all the boxes of the other (second) in all possible ways with the following rules:

(a) The boxes of the multiplying (second) tableau must be attached to the fixed (first) one row by row, starting with the uppermost one.

(b) The resulting tableau at all stages must again be a proper Young tableau. This means that the number of boxes in any row must not be increasing as we go from the upper to the lower row.

(c) The two boxes from the same row can never be placed in the same column.

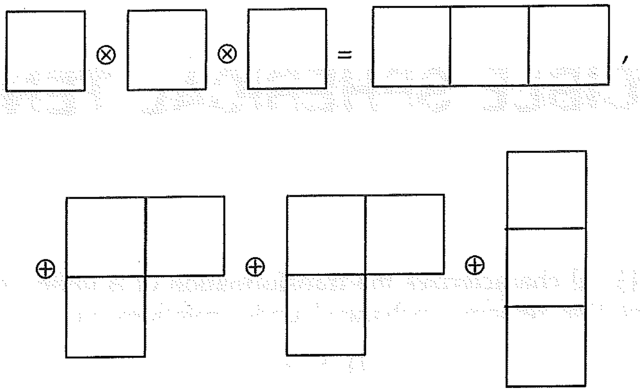
(d) The number of rows in the resulting diagram must not exceed N. Every column with N boxes, being singlet, must be dropped out of the final diagram.

(e) On a path through each row from the right, and moving from top to bottom through the entire diagram, at any point on the path the number of boxes encountered containing the number n must be less than or equal to the number of boxes encountered containing the number $n - 1$.

(f) The number of boxes in a row does not decrease from left to right. The number of boxes in a column increases from top to bottom.

PROBLEMS

Show that



$$N \otimes N \otimes N = \frac{1}{6}\,N(N+1)(N+2) \oplus \frac{1}{3}\,N(N+1)(N-1) \oplus \frac{1}{3}\,N(N+1)(N-1) \oplus \frac{1}{6}\,N(N-1)(N-2).$$

Write the answer for $SU(2)$, $SU(3)$, and $SU(6)$.

EXAMPLES IN $SU(3)$

Show that

$$3 \otimes \bar{3} = 1 \oplus 8$$

Now consider $8 \otimes 8$ and show that it is equal to

$$27 \oplus 10 \oplus 10 \oplus 8 \oplus 8 \oplus 1$$

Appendix B

IRREDUCIBLE SPHERICAL TENSORS

The commutator $[\mathbf{J}, A]$ characterizes the transformation of A under rotations. A scalar S, for example, is one that remains unchanged under rotations, i.e.

$$[\mathbf{J}, S] = 0. \tag{B.1}$$

A vector operator is a set of three operators V_1, V_2, V_3 which transform under rotations in the same way as the components of the position vector do. It is easily verified that this means

$$[J_i, V_j] = \varepsilon_{ijk} V_k. \tag{B.2}$$

Scalar and vector operators are particularly simple types. We shall mainly be concerned with vector operators. It is useful to define the spherical components of $\mathbf{V}$ as

$$\left.\begin{aligned} V_1^{(1)} &= -\frac{1}{\sqrt{2}}(V_x + iV_y), \\ V_0^{(1)} &= V_z, \\ V_{-1}^{(1)} &= \frac{1}{\sqrt{2}}(V_x - iV_y). \end{aligned}\right\} \tag{B.3}$$

The set of three operators satisfies the commutation relations

$$\left.\begin{aligned} [J_z, V_q] &= qV_q, \\ [J_+, V_q] &= \sqrt{(1-q)(2+q)}\, V_{q+1}, \\ [J_-, V_q] &= \sqrt{(1+q)(2-q)}\, V_{q-1}. \end{aligned}\right\} \tag{B.4}$$

where $V_q = 0$ if $q \neq 0, \pm 1$. More generally, we may define an irreducible spherical tensor operator of rank (k) as a set of $2k + 1$ operators $T_q^{(k)}$ $(q = k, k-1, k-2, \ldots, -k)$ which transform according to

$$[J_z, T_q^{(k)}] = qT_q^{(k)},$$

$$[J_\pm, T_q^{(k)}] = \sqrt{(k \mp q)(k \pm q + 1)}\, T_{q\pm 1}^{(k)}.$$

In other words, the $T_q^{(k)}$ transforms like an object with spin (k). The components of an irreducible tensor operator may be put in one-to-one correspondence with the basis states $|jm\rangle$ of an irreducible representation. Every operator can be written as a linear combination of irreducible tensor operators.

B.1 Applications of Irreducible Tensor Operator

Wigner-Eckart theorem

$$\langle j_2 m_2 | T_q^{(k)} | j_1 m_1 \rangle = \langle j_2 m_2 | j_1 k;\ m_1 q \rangle \langle j_2 || T_q^{(k)} || j_1 \rangle. \tag{B.5}$$

A matrix element of an irreducible spherical tensor operator can always be separated into a product of a Clebsch-Gordan coefficient and a "reduced matrix element" which depends only on the irreducible representations and the nature of the tensor involved.

The Wigner-Eckart theorem, thus, implies that all the matrix elements corresponding to various values of m_1, m_2, q are proportional to each other, the constant of proportionality depending only on the order (k) of the tensor operator and the total angular momentum of the two states. Thus the Wigner-Eckart theorem allows us to factor the physical content of a given operator from the purely geometrical properties describing how it transforms under rotations. Consider, as an example of the application of the Wigner-Eckart theorem, the decays of $i = 3/2$ (1238) resonances into $i = 1$, pion and $i = 1/2$ nucleon, namely,

$$\Delta^{++} \to \pi^+ p, \quad \Delta^+ \to \pi^0 p, \quad \Delta^+ \to \pi^+ n,$$
$$\Delta^0 \to \pi^- p, \quad \Delta^0 \to \pi^0 n, \quad \Delta^- \to \pi^- n.$$

This is an example of an $i = 3/2$ state a, decaying to $i = 1$, b, and, $i = 1/2$ c, states $(a \to b + c)$. We need to evaluate the matrix element

$$\langle bc | O | a \rangle, \tag{B.6}$$

where O is the operator responsible for the transition. We shall assume that O is a scalar under isospin transformation. We need to calculate

$$\langle i_b,\ i_{b3};\ i_c,\ i_{c3} | O | i_a,\ i_{a3} \rangle = \langle i_b\ i_{b3};\ i_c,\ i_{c3} | i_a,\ i_{a3} \rangle \times \langle i_b i_c || O || i_a \rangle. \tag{B.7}$$

If we write

$$\langle 1, \frac{1}{2} || O || \frac{3}{2} \rangle = M, \tag{B.8}$$

then we have the following results

$$\begin{array}{ll}
\Delta^{++} \to \pi^+ p, & M, \\
\Delta^+ \to \pi^0 p, & \sqrt{2/3}\,M, \\
\quad\ \ \to \pi^+ n, & \sqrt{1/3}\,M, \\
\Delta^0 \to \pi^- p, & \sqrt{1/3}\,M, \\
\quad\ \ \to \pi^0 n, & \sqrt{2/3}\,M, \\
\Delta^- \to \pi^- n, & M.
\end{array}$$

We see that all decays are expressed in terms of one matrix element, the coupling constant.

Appendix C

NOETHER'S THEOREM

C.1 Noether's Theorem

The connection between conservation laws and invariance principle was made in the 1920s by an eminent mathematician, Emmy Noether (1882–1935), working in Gottingen, Germany. She observed that the law of the conservation of momentum implies invariance of the laws of physics to a spatial displacement. Noether's theorem asserts that if all the values of the spatial coordinate x are replaced by $x + A$ (i.e. if the entire system under study is shifted by A units in the x-direction), then the x-component of total momentum is conserved in all reactions. There is no preferred location. The laws of physics, the results of all possible experiments, do not depend on the absolute location in space of the laboratory. Thus, all the equations describing all the events must be so structured that when x is changed by adding A, the value of A simply cancels everywhere.

In a similar way, Noether showed that the conservation of angular momentum is related to the symmetry of direction in space (isotropy). All directions are the same, and if every direction in physical system is rotated by, say, B degrees the laws of physics do not change; they are invariant and B drops out of all equations. Non-spatial conservation laws are also known and they teach us about new symmetries.

C.1.1 Statement of Noether's Theorem

The invariance of the action integral under continuous transformations leads to certain conservation laws for the fields associated with the Lagrangian.

C.1.2 Proof

Let us consider the transformations for the co-ordinates

$$x^\lambda \to x'^\lambda = x^\lambda + \delta x^\lambda, \tag{C.1}$$

where δx^λ represents infinitesimal change (transformation). Similarly, let us take the transformation for the field function as

$$\phi_\sigma(x) \to \phi'_\sigma(x') = \phi_\sigma(x) + \delta\phi_\sigma(x), \tag{C.2}$$

where $\delta\phi_\sigma(x)$ represents the change in the field function, arising both from the change in the form of the function and change in its argument x. We may write the variation of the action integral as

$$\delta A = \delta \int d^4x \mathcal{L} = \int \delta(d^4x)\mathcal{L} + \int d^4x \delta\mathcal{L} = 0. \tag{C.3}$$

$\mathcal{L}$ is the Lagrangian density.

$$d^4x \to d^4x',$$

where

$$d^4x' = \left(1 + \frac{\partial}{\partial x^\lambda}\delta x^\lambda\right) d^4x. \tag{C.4}$$

Thus,

$$\delta(d^4x) = d^4x \frac{\partial(\delta x^\lambda)}{\partial x^\lambda}. \tag{C.5}$$

So,

$$\delta A = \int d^4x \left(\delta\mathcal{L} + \mathcal{L}\frac{\partial(\delta x^\lambda)}{\partial x^\lambda}\right) = 0. \tag{C.6}$$

Now,

$$\delta\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \bar{\delta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^\lambda}\delta x^\lambda, \tag{C.7}$$

where $\bar{\delta}\mathcal{L}$ represents intrinsic variation in the form of the Lagrangian. We may thus write

$$\delta A = \int d^4x \left[\bar{\delta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial x^\lambda}\delta x^\lambda + \mathcal{L}\frac{\partial(\delta x^\lambda)}{\partial x^\lambda}\right] = 0, \tag{C.8}$$

$$\int d^4x \left[\bar{\delta}\mathcal{L} + \frac{\partial}{\partial x^\lambda}(\mathcal{L}\delta x^\lambda)\right] = 0. \tag{C.9}$$

Since $\mathcal{L}$ is a function of ϕ_σ and $\partial_\mu\phi_\sigma$, we have

$$\bar{\delta}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_\sigma}\bar{\delta}\phi_\sigma + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\bar{\delta}(\partial_\mu\phi_\sigma). \tag{C.10}$$

Using the Euler-Lagrange equations, we may write

$$\bar{\delta}\mathcal{L} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\bar{\delta}\phi_\sigma + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\partial_\mu(\bar{\delta}\phi_\sigma) = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\bar{\delta}\phi_\sigma\right], \tag{C.11}$$

substitution of which gives for the change in action as

$$\delta A = \int d^4x \left\{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\bar{\delta}\phi_\sigma\right] + \frac{\partial}{\partial x^\lambda}(\mathcal{L}\delta x^\lambda)\right\}. \tag{C.12}$$

Now,

$$\delta\phi_\sigma = \phi'_\sigma(x') - \phi_\sigma(x) = \bar{\delta}\phi_\sigma + \frac{\partial\phi_\sigma}{\partial x^\lambda}\,\delta x^\lambda,$$

where

$$\bar{\delta}\phi_\sigma = \phi'_\sigma(x) - \phi_\sigma(x) \tag{C.13}$$

so that

$$\bar{\delta}\phi_\sigma = \delta\phi_\sigma - \frac{\partial\phi_\sigma}{\partial x^\lambda}\,\delta x^\lambda \tag{C.14}$$

for which

$$\delta A = \int d^4x \left\{ \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)} (\delta\phi_\sigma - \frac{\partial\phi_\sigma}{\partial x^\lambda}\,\delta x^\lambda) \right] + \partial_\mu(\delta_{\mu\lambda}\,\mathcal{L}\delta x^\lambda) \right\}$$

$$= \int d^4x\, \partial_\mu \Bigg[\bigg[\bigg\{ \delta^\mu_\lambda\,\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\frac{\partial\phi_\sigma}{\partial x^\lambda} \bigg\} \delta x^\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\,\delta\phi_\sigma \Bigg] = 0. \tag{C.15}$$

Since the volume element is arbitrary, we get the continuity equation

$$\frac{\partial}{\partial x^\mu} f^\mu = 0, \tag{C.16}$$

where the conserved current is

$$f^\mu = \Bigg[\bigg[\bigg\{ \delta^\mu_\lambda\,\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\frac{\partial\phi_\sigma}{\partial x^\lambda} \bigg\} \delta x^\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\,\delta\phi_\sigma \Bigg]. \tag{C.17}$$

The constant of motion C is obtained by integrating the time component of the conserved current. C is thus

$$C = \int d^3x f^0. \tag{C.18}$$

It is easy to see that C is a constant of motion.

$$\frac{d}{dt}C = \frac{d}{dt}\int d^3x f^0 = \int d^3x \frac{d}{dt} f^0 = -\int d^3x \frac{d}{dx^k} f^k. \tag{C.19}$$

Being a three-dimensional divergence, the integrand on the right-hand side can be replaced by a surface integral with the aid of Gauss theorem and will be zero.

C.1.3 Importance of Noether's Theorem

This theorem tells us when conserved quantities will exist. If there is a symmetry, we know there is a conserved quantity, and if there is a conserved quantity, we know there must be a symmetry. Noether's theorem helps us in relating observed symmetries and conservation laws to the structure of the physical systems.

C.1.4 Applications

Consider the space-time translation as the first example.

$$x^\lambda \to x^\lambda + a^\lambda, \qquad a^\lambda \text{ is constant} \tag{C.20}$$

The constant of motion C is given by

$$C = \int d^3x f^0 = \left[\left\{\delta^0_\lambda \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial^0\phi_\sigma)}\frac{\partial \phi_\sigma}{\partial x^\lambda}\right\} a^\lambda\right] = P^\lambda a^\lambda. \tag{C.21}$$

P^λ is the four momentum of the field. Since a^λ is arbitrary the conservation of C means the conservation of the four momentum P^λ, a result we are familiar with.

C.2 Gauge Transformations

As another application of Noether theorem, let us consider gauge transformations of the first kind. Here,

$$x^\lambda \to x'^\lambda = x^\lambda \tag{C.22}$$

so that

$$\delta x^\lambda = 0,$$

$$\phi_\sigma(x) \to \phi'_\sigma(x') = \phi'_\sigma(x), \tag{C.23}$$

and

$$\phi'_\sigma = e^{i\alpha}\phi_\sigma = \phi_\sigma + i\alpha\phi_\sigma \tag{C.24}$$

for an infinitesimal transformation, Hence

$$\delta\phi_\sigma = i\alpha\phi_\sigma. \tag{C.25}$$

Also,

$$\phi^*_\sigma \to \phi^{*\prime}_\sigma = e^{-i\alpha}\phi^*_\sigma = \phi^*_\sigma - i\alpha\phi^*_\sigma, \tag{C.26}$$

and, therefore,

$$\delta\phi^*_\sigma = -\,i\alpha\phi^*_\sigma. \tag{C.27}$$

Substituting these into the expression for the conserved current f_μ, we obtain

$$f^\mu = i\alpha\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_\sigma)}\phi_\sigma - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*_\sigma)}\phi^*_\sigma\right]. \tag{C.28}$$

From (C.21), we can immediately obtain the constant of motion by integrating the time component f_0 of the conserved current.

$$C = \int d^3x\, i\alpha\left[\frac{\partial \mathcal{L}}{\partial(\dot{\phi}_\sigma)}\phi_\sigma - \frac{\partial \mathcal{L}}{\partial(\dot{\phi}^*_\sigma)}\phi^*_\sigma\right] = -\,i\alpha\int d^3x\,[\pi^*_\sigma\phi^*_\sigma - \pi_\sigma\phi_\sigma] = -\,i\alpha Q. \tag{C.29}$$

The quantity Q is called a charge and is the generator of the gauge transformation.

Appendix D

RELATIVISTIC KINEMATICS

The contravariant four momentum is written as $P^\mu = (P^0, \mathbf{P})$ with $P^0 = E/c = P_0$. The scalar product $P^\mu P_\mu$ is a Lorentz invariant and has the value M^2c^2 in all frames. In general, $A^\mu B_\mu = A^0B^0 - \mathbf{A} \cdot \mathbf{B}$ is invariant.

Consider a decay $A \to B + C$ with masses M, m_1 and m_2 of A, B, and C respectively. From the conservation of energy and momentum, we have $P = p_1 + p_2$, where P, p_1 and p_2 are the four momenta of A, B and C respectively. Now consider the invariant

$$p_2^2 = (P - p_1)^2 = P^2 + p_1^2 - 2P \cdot p_1, \tag{D.1}$$

i.e.

$$m_2^2c^2 = M^2c^2 + m_1^2c^2 - 2P \cdot p_1, \tag{D.2}$$

holds in all frames. Take the rest frame of the decaying particle. That is, take $\mathbf{P} = 0$. We have

$$m_2^2c^2 = M^2c^2 + m_1^2c^2 - 2ME_1 \text{ since } (P \cdot p_1 = Mc^2E_1/c^2 - 0). \tag{D.3}$$

So the energy E_1 of the particle 1 in the rest frame of the decaying particle is given by

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2. \tag{D.4}$$

Similarly,

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2. \tag{D.5}$$

The kinetic energy of particle 1 is given by

$$\begin{aligned} T_1 = E_1 - m_1c^2 &= \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 - m_1c^2. \\ &= \frac{M^2 + m_1^2 - m_2^2 - 2Mm_1}{2M} c^2 = \frac{(M - m_1)^2 - m_2^2}{2M} c^2 \\ &= \frac{(M - m_1 - m_2)(M - m_1 + m_2)}{2M} c^2 = \Delta Mc^2 \left[\frac{(M - m_1 + m_2)}{2M}\right] \\ &= \Delta Mc^2 \left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right). \end{aligned} \tag{D.6}$$

For example, consider the decay $\pi \to \mu + \nu_\mu$ with masses 139.6, 105.7 and 0 MeV for π, μ and ν respectively. The kinetic energy of the muon is

$$T_\mu = 33.9\left(1 - \frac{105.7}{139.6} - \frac{33.9}{2 \times 139.6}\right) = 4.1\ \text{MeV}. \tag{D.7}$$

D.1 Relation Between Lab and CM Frames

Let us consider a system of two particles. It may be a part of a reaction, i.e. it may be the initial or the final state of a reaction. Let p_1 and p_2 be their four momenta in frame S and p_1' and p_2' in S'. $(p_1 + p_2) \cdot (p_1 + p_2) = (p_1' + p_2') \cdot (p_1' + p_2')$ is an invariant. Choose S to be the lab and S' to be the Centre of Mass (CM) frame. Then $\mathbf{p}_2 = 0$ and let $\mathbf{p}_1 = \mathbf{p}$. Also, $\mathbf{p}' = \mathbf{p}_1' + \mathbf{p}_2' = 0$. The above equation gives

$$\frac{(E_1 + E_2)^2}{c^2} - \mathbf{p}^2 = \frac{E'^2}{c^2} \quad \text{with} \quad E_2 = m_2c^2 \tag{D.8}$$

or

$$(E_1 + m_2c^2)^2 - c^2\mathbf{p}^2 = E'^2, \tag{D.9}$$

So,

$$E'^2 = E_1^2 - c^2\mathbf{p}^2 + m_2^2c^4 + 2E_1m_2^2c^4 = m_1^2c^4 + m_2^2c^4 + 2E_1m_2c^2, \tag{D.10}$$

$$E' = (m_1^2c^4 + m_2^2c^4 + 2E_1m_2c^2)^{1/2} \tag{D.11}$$

constitute the total energy in the centre of mass system, whereas

$$E_1 = \frac{E'^2 - m_1^2c^4 - m_2^2c^4}{2m_2c^2} \tag{D.12}$$

is the lab energy of particle 1. If we want to obtain the CM energies E_1' and E_2', we shall start with the invariant

$$p_1 \cdot (p_1 + p_2) = p_1' \cdot (p_1' + p_2') = E_1(E_1 + m_2c^2) - c^2\mathbf{p}^2 = E_1'E' \tag{D.13}$$

yielding

$$E_1^2 - c^2\mathbf{p}^2 + E_1m_2c^2 = E_1'E' = m_1^2c^4 + E_1m_2c^2 \tag{D.14}$$

so that

$$E_1' = \frac{m_1^2c^4 + E_1m_2c^2}{E'} = \frac{E'^2 + m_1^2c^4 - m_2^2c^4}{2E'}, \tag{D.15}$$

where, to write the right-hand side, we have used (D.10). Similarly, starting with the invariant

$$p_2 \cdot (p_1 + p_2) = p_2' \cdot (p_1' + p_2'), \tag{D.16}$$

we obtain

$$m_2c^2\,(E_1 + m_2c^2) = E_2'E', \tag{D.17}$$

giving

$$E_2' = \frac{m_2c^2(E_1 + m_2c^2)}{E'} = \frac{E'^2 - m_1^2c^4 + m_2^2c^4}{2E'}. \tag{D.18}$$

E_2' can also be obtained as follows:

$$E_2' = E' - E_1' = E' - \frac{E'^2 + m_1^2c^4 - m_2^2c^4}{2E'} = \frac{E'^2 + m_2^2c^4 - m_1^2c^4}{2E'}$$

$$= \frac{m_1^2c^4 + m_2^2c^4 + 2E_1m_2c^2 + m_2^2c^4 - m_1^2c^4}{2E'} = \frac{m_2c^2(E_1 + m_2c^2)}{E'} \tag{D.19}$$

In the special case, when $m_1 = m_2$,

$$E_1' = E_2' = \frac{1}{2}E'. \tag{D.20}$$

The direction of **p** and **p**′ is the same. The magnitudes p and p' are related as:

$$c^2p'^2 = E_1'^2 - m_1^2c^4 = \frac{1}{4E'^2}[E'^2 + m_1^2c^4 - m_2^2c^4]^2 - m_1^2c^4$$

$$= \left[\frac{1}{2E'}(E'^2 + m_1^2c^4 - m_2^2c^4) + m_1c^2\right]\left[\frac{1}{2E'}(E'^2 + m_1^2c^4 - m_2^2c^4) - m_1c^2\right]$$

$$= \frac{1}{4E'^2}(E' + m_1c^2 + m_2c^2)(E' + m_1c^2 - m_2c^2)(E' - m_1c^2 + m_2c^2)(E' - m_1c^2 - m_2c^2). \tag{D.21}$$

Now,

$$c^2p^2 = E_1^2 - m_1^2c^4 = \frac{1}{4m_2^2c^4}(E'^2 - m_1^2c^4 - m_2^2c^4)^2 - m_1^2c^4.$$

$$= \frac{1}{4m_2^2c^4}(E' + m_1c^2 + m_2c^2)(E' + m_1c^2 - m_2c^2)(E' - m_1c^2 + m_2c^2)(E' - m_1c^2 - m_2c^2). \tag{D.22}$$

So,

$$\frac{c^2p'^2}{c^2p^2} = \frac{(m_2c^2)^2}{E'^2}, \tag{D.23}$$

or

$$\frac{p'}{p} = \frac{m_2c^2}{E'}. \tag{D.24}$$

D.2 Centre of Mass Frame

Let $\mathbf{v}_{CM}$ be the velocity of the Centre of Mass (CM) frame in laboratory (lab), and γ_{CM} be the corresponding γ. Then,

$$\mathbf{p}_2' = \gamma_{\mathrm{CM}} m_2 \mathbf{v}_{\mathrm{CM}} = -\mathbf{p}_1' = -\mathbf{p}', \tag{D.25}$$

$$E_2' = \gamma_{\mathrm{CM}} m_2 c^2. \tag{D.26}$$

So,

$$\mathbf{v}_{\mathrm{CM}} = \frac{c^2\mathbf{p}'}{E_2'}. \tag{D.27}$$

We want to find $\mathbf{v}_{\mathrm{CM}}$ and γ_{CM} in terms of the lab parameters. We have seen before that

$$p' = \frac{m_2 c^2 p_{\mathrm{lab}}}{E'}, \quad E_2' = \frac{m_2^2 c^4 + E_1 m_2 c^2}{E'} \tag{D.28}$$

so that

$$\frac{p'}{E_2'} = \frac{m_2 c^2 p_{\mathrm{lab}}}{E'} \times \frac{E'}{m_2^2 c^4 + E_1 m_2 c^2} = \frac{p_{\mathrm{lab}}}{(E_1 + m_2 c^2)} \tag{D.29}$$

and hence

$$\mathbf{v}_{\mathrm{CM}} = \frac{c^2\mathbf{p}'}{E_2'} = \frac{p_{\mathrm{lab}}}{(E_1 + m_2 c^2)}. \tag{D.30}$$

Further, since, $E_2' = \gamma_{\mathrm{CM}} m_2 c^2$, we have

$$\gamma_{\mathrm{CM}} = \frac{E_{\mathrm{lab}} + m_2 c^2}{E'}. \tag{D.31}$$

D.3 Threshold

Consider

$$1 + 2 \rightarrow 3 + 4 + 5 \ldots,$$

$$\Delta M = (m_3 + m_4 + m_5 + \ldots) - (m_1 + m_2). \tag{D.32}$$

For positive ΔM, the reaction will not occur unless there is at least T_{th} kinetic energy incidently. In the centre of mass system,

$$E_{\mathrm{th}}' = m_1 c^2 + m_2 c^2 + \Delta M c^2 \tag{D.33}$$

is the total energy of the final state. The total CM energy is related to the energy E_1 of the particle 1 in the lab through

$$E' = (m_1^2 c^4 + m_2^2 c^4 + 2 E_1 m_2 c^2)^{1/2}. \tag{D.34}$$

So,

$$E_{\mathrm{th}}' = (m_1^2 c^4 + m_2^2 c^4 + 2 E_{1\mathrm{th}} m_2 c^2)^{1/2}. \tag{D.35}$$

Now defining

$$E_{1\mathrm{th}} = m_1 c^2 + T_{\mathrm{th}}, \tag{D.36}$$

we readily obtain

$$T_{\mathrm{th}} = \Delta M c^2 \left(1 + \frac{m_1}{m_2} + \frac{\Delta M}{2 m_2}\right). \tag{D.37}$$

D.3.1 Weak Interactions

$$\frac{1^+}{2} \to \frac{1^+}{2} + 0^-. \tag{D.38}$$

For a weak decay of a baryon into a baryon and a meson, the matrix element is a mixture of a scalar and a pseudoscalar as parity is not conserved in the weak interaction. The matrix element, therefore, can be represented as

$$\bar{u}(p_b)\,[A + B\gamma_5]u(p_a). \tag{D.39}$$

A is parity violating amplitude, whereas B is parity conserving amplitude. A is also known as the s-wave amplitude and B the p-wave amplitude.

D.4 πl_2 Decays

$$\pi \to l + \bar{\nu}_l. \tag{D.40}$$

Invoking Lorentz covariance, the matrix element of the weak current between vacuum and one pion state is given by

$$\langle 0\,|\,j_\mu\,|\,\pi(p)\rangle = if_\pi\, p_\mu. \tag{D.41}$$

It is because the only four vector available for the process is the four momentum of the pion. Further, since the intrinsic parity of the pion is odd, only the axial vector part of the weak current will contribute to the matrix element.

D.4.1 Electromagnetic

We have seen in Chapter 3 that the matrix element of vector current (electromagnetic or weak) between two nucleon states can, on the basis of Lorentz covariance, can be written as

$$\langle P'|j^\mu|P\rangle = \bar{u}(P')(F_1\gamma^\mu + F_3 q^\mu + F_2 i\sigma^{\mu\nu}q_\nu)u(P). \tag{D.42}$$

D.5 Trace Formulae

$$\mathrm{Tr}(1) = 4, \qquad \mathrm{Tr}(\gamma^\mu) = 0, \qquad \mathrm{Tr}(\gamma^5) = 0. \tag{D.43}$$

The trace of a product of odd number of γ's vanishes. For product of even number of γ's, the following relations are useful:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \tag{D.44}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0, \tag{D.45}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}), \tag{D.46}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^5) = 4i\varepsilon^{\mu\nu\lambda\sigma}. \tag{D.47}$$

Appendix E

DECAY RATES AND CROSS-SECTIONS

E.1 Decay Rates and Cross-sections

For the decay rate $\Gamma(i \rightarrow f)$, we have

$$\Gamma(i \rightarrow f) = \frac{(2\pi)^4}{(2\pi)^{3n} 2^n 2E_i} \int \prod_{j=1}^{n} \frac{d^3 p_j}{E_j} \delta^4 (p_i - \sum_j p_j) |\langle f | T | i \rangle|^2, \tag{E.1}$$

whereas for a collision process

$$i \equiv a + b \rightarrow f, \tag{E.2}$$

the cross-section is given by

$$d\sigma = \frac{(2\pi)^4 \delta^4 (p_a + p_b - p_f)}{(2\pi)^{3n} 4 E_a E_b v} |T_{fi}|^2 \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j}. \tag{E.3}$$

E.1.1 Phase Space Integral for $a(\mathbf{p}) \rightarrow b(\mathbf{p}') + c(\mathbf{q})$

For the case of a decay of a particle into two particles, the phase integral is simple.

$$\int \delta^4 (p - p' - q) d^3 p' d^3 q \, |M|^2 \tag{E.4}$$

Further, because of the delta function, all variables contribute to the integral at fixed values. The matrix element of the decay is, therefore, a constant and hence the integral does not depend on the nature of the decay and involves only the kinematical factors. Let us consider

$$I = \int \delta^4 (p - p' - q) d^3 p' d^3 q = \int \delta^3 (\mathbf{p} - \mathbf{p}' - \mathbf{q}) \delta(p_0 - p'_0 - q_0) d^3 p' d^3 q$$

$$= \int \delta(p_0 - p'_0 - q_0) d^3 q, \tag{E.5}$$

with

$$\mathbf{p} = \mathbf{p}' + \mathbf{q}. \tag{E.6}$$

We evaluate this integral in the rest frame of the decaying particle which is the same as the centre of mass frame under consideration. Then $\mathbf{p} = 0$, $p_0 = M_a$; $\mathbf{p}' = -\mathbf{q}$. Further,

$$d^3q = q^2 dq d\Omega = |\mathbf{q}|\, q_0 dq_0 d\Omega. \tag{E.7}$$

The above integral then becomes

$$I = \int \delta(p_0 - p_0' - q_0)\,|\mathbf{q}|\, q_0 dq_0 d\Omega$$

$$= \int \delta[p_0 - \{q_0^2 - m_c^2 + m_b^2\}^{1/2} - q_0]\,|\mathbf{q}|\, q_0 dq_0\, d\Omega. \tag{E.8}$$

To evaluate this integral, we make use of the identity

$$\delta(f(x)) = -\frac{\delta(x - x_0)}{f'(x)|_{x_0}}, \tag{E.9}$$

where x_0 is the zero of the function $f(x)$.

$$f(x) = p_0 - \{x^2 - m_c^2 + m_b^2\}^{1/2} - x, \tag{E.10}$$

$$f'(x)|_{x_0} = -\frac{1}{2}\{x_0^2 - m_c^2 + m_b^2\}^{-1/2}\, 2x_0 - 1$$

$$= -\frac{q_0}{p_0'} - 1 = -\frac{m_a}{p_0'}, \tag{E.11}$$

where x_0 is obtained by setting

$$m_a - \{x_0^2 - m_c^2 + m_b^2\}^{1/2} - x_0 = 0, \tag{E.12}$$

giving

$$(m_a - x_0)^2 = x_0^2 - m_c^2 + m_b^2, \tag{E.13}$$

and so

$$x_0 = \frac{m_c^2 - m_b^2 + m_a^2}{2m_a}. \tag{E.14}$$

So the phase space integral is

$$\int \delta(q_0 - x_0) dq_0\, |\mathbf{q}|\, q_0 \frac{p_0'}{m_a}\, d\Omega = |\mathbf{q}|\, q_0 \frac{p_0'}{m_a}\, d\Omega, \tag{E.15}$$

where in (E.15), $q_0 = x_0$, and the formula for the decay rate is

$$\Gamma(a \to b + c) = \frac{|\mathbf{q}|}{32\pi^2 m_a^2} \int d\Omega \sum_{\text{spin}} |M|^2. \tag{E.16}$$

E.2 Three Particle Phase Space

The invariant three particle phase space factor is

$$\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}\, \delta^4(P_i - P_f)$$

where in the rest frame of the decaying particle,

$$\delta^4(P_i - P_f) = \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\ \delta(E_1 + E_2 + E_3 - M).$$

Integration over $\mathbf{p}_3$ gives

$$\int \frac{1}{8E_1E_2E_3}\, p_1^2\, dp_1\, d\Omega_1\, p_2^2\, dp_2\, d\Omega_2\ \delta(E_1 + E_2 + E_3 - M).$$

By fixing the direction of $\mathbf{p}_1$ and choosing it as the z-axis, we integrate over the polar angles of $\mathbf{p}_2$ which means angle θ between $\mathbf{p}_1$ and $\mathbf{p}_2$. Then we integrate over the direction of $\mathbf{p}_1$ to get

$$\pi^2 \int \frac{p_1^2 dp_1}{E_1}\, \frac{p_2^2 dp_2}{E_2}\, \frac{d(d\cos\theta)}{E_3}\ \delta(E_1 + E_2 + E_3 - M).$$

Using the relations

$$E_1^2 = p_1^2 + m_1^2, \qquad E_2^2 = p_2^2 + m_2^2,$$

we get

$$E_1 dE_1 = p_1 dp_1, \qquad E_2 dE_2 = p_2 dp_2.$$

Further,

$$E_3^2 = p_3^3 + m_3^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\theta_{12} + m_3^2$$

so that for fixed p_1 and p_2,

$$E_3 dE_3 = p_1 p_2 d(\cos\theta).$$

Thus the phase factor becomes

$$\int dE_1 dE_2 dE_3 \delta(E_1 + E_2 + E_3 - M) = \int dE_1 dE_2.$$

If we include the square of the matrix element, the decay rate will be proportional to

$$\int dE_1 dE_2\, |M_{fi}|^2.$$

In particular, if we consider the decay of τ into three pions, the decay rate is given by

$$\omega = \frac{4\pi}{(2s_a + 1)\, m_a} \int dE_1\, dE_2 \sum_{\text{spin}} |M_{\tau\to 3\pi}|^2.$$

In order to evaluate this integral, we need to find the domain of integration. The plot of the experimental data on energy distribution can then give us information about the nature of the matrix element and also about the spin of the decaying particle. The plot is known as Dalitz plot and has been used to find the spin of τ and resonances like ω.

Appendix F

GELL-MANN MATRICES

The unitary symmetry $SU(3)$ has $3^2 - 1 = 8$ generators. In the fundamental representation they are traceless 3×3 matrices. Two of these can be taken as diagonal. We take λ_3 and λ_8 as diagonal. In the canonical representation

$$U = e^{i\lambda_k \alpha_k}, \qquad (k = 1, 2, 3, \ldots, 8) \tag{F.1}$$

where $\lambda_k/2$ are the group generators with the choice for the Gell-Mann matrices λ_k

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{F.2}$$

The generators satisfy the Lie algebra

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = if_{ijk}\frac{\lambda_k}{2}, \tag{F.3}$$

where f_{ijk}, the structure constants are completely antisymmetric under exchange of indices. The nonvanishing f_{ijk} are

$$\begin{aligned} f_{123} &= 1, \\ f_{147} &= f_{246} = f_{257} = f_{345} = 1/2, \\ f_{156} &= f_{367} = -1/2, \\ f_{458} &= f_{678} = \sqrt{3}/2. \end{aligned}$$

The generators are traceless and are normalized as

$$\mathrm{Tr}\left(\frac{\lambda_i}{2}\cdot\frac{\lambda_j}{2}\right)=\frac{1}{2}\,\delta_{ij}\,.$$

The λ_i also satisfy the anticommutation relations

$$\left\{\frac{\lambda_i}{2},\frac{\lambda_j}{2}\right\}=\frac{1}{3}\,\delta_{ij}+d_{ijk}\left(\frac{\lambda_k}{2}\right).$$

The d_{ijk} are symmetric under exchange of indices. Those with nonzero values are

$$\begin{aligned}
d_{118} &= d_{228} = d_{338} = -\,d_{888} = 1/\sqrt{3}\,,\\
d_{146} &= d_{157} = d_{256} = d_{344} = d_{355} = 1/2,\\
d_{247} &= d_{366} = d_{377} = -\,1/2,\\
d_{448} &= d_{558} = d_{668} = -\,d_{778} = -\,\frac{1}{2\sqrt{3}}\,.
\end{aligned}$$

We introduce the notation

$$F_k=\lambda_k/2.$$

We then construct two diagonal operators

$$I_3=F_3,\qquad Y=\frac{2}{\sqrt{3}}\,F_8,$$

and the remaining six generators

$$I_\pm=F_1\pm iF_2,\qquad V_\pm=F_4\pm iF_5,\qquad U_\pm=F_6\pm iF_7.$$

The commutation relations and significance of $I_\pm$, $V_\pm$ and $U_\pm$ operators have been discussed in Chapter 4.

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INTRODUCTION TO PARTICLE PHYSICS

M.P. KHANNA

This comprehensive and well-organized book focusses on the phenomenological aspects of Particle Physics. It strikes a fine balance between those texts that require sophisticated mathematical physics and those that are too elementary. For, unlike in many books on the subject, which give prominence to gauge theories, the attempt here is to lay stress on phenomenology — an aspect that needs exposure among students of high energy physics.

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Designed as a one-semester/one-year course on Particle Physics at the postgraduate level and as a first reading for research students, the book should prove both useful and stimulating.

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